THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE two hundred and third regular meeting of the Society was held in New York City on Saturday, April 26, 1919, extending through the usual morning and afternoon sessions. This being the first eastern meeting since October, the attendance was large, indicating, as it may be hoped, a revival of the conditions preceding the war. The following sixty-seven members were present:

Professor R. C. Archibald, Professor R. A. Arms, Dr. Charlotte C. Barnum, Professor Susan R. Benedict, Professor G. D. Birkhoff, Professor B. H. Camp, Professor F. N. Cole, Professor Louise D. Cummings, Professor C. H. Currier, Dr. Mary F. Curtis, Dr. Tobias Dantzig, Dr. J. V. DePorte, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor A. B. Frizell, Mr. T. C. Fry, Professor W. V. N. Garretson, Professor O. E. Glenn, Professor W. C. Graustein, Dr. T. H. Gronwall, Professor C. C. Grove, Professor C. O. Gunther, Captain Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor O. D. Kellogg, Professor C. J. Keyser, Dr. E. A. T. Kircher, Dr. J. R. Kline, Dr. K. W. Lamson, President E. O. Lovett, Professor James Maclay, Professor L. C. Mathewson, Professor R. L. Moore, Professor F. M. Morgan, Professor Frank Morley, Professor G. W. Mullins, Dr. F. D. Murnaghan, Mr. L. S. Odell, Mr. George Paaswell, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor Susan M. Rambo, Professor H. W. Reddick, Professor R. G. D. Richardson, Dr. J. F. Ritt, Professor E. D. Roe, Jr., Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Dr. J. M. Stetson, Professor H. D. Thompson, Mr. H. S. Vandiver, Major Oswald Veblen, Mr. H. E. Webb, Dr. Mary E. Wells, Professor H. S. White, Mr. J. K. Whittemore, Dr. C. E. Wilder, Miss Ella C. Williams, Dr. Emily C. Williams, Professor Ruth G. Wood, Professor J. W. Young.

President Morley occupied the chair, being relieved by Professor Kasner. The Council announced the election of the following persons to membership in the Society: Mr. N. W. Akimoff, Philadelphia, Pa.; Dr. Tobias Dantzig, Columbia

University; Mr. A. C. Maddox, Guthrie, Okla.; Mr. Montford Morrison, Chicago, Ill.; Professor Ganesh Prasad, Central Hindu College, Benares, India; Mr. F. M. Weida, State University of Iowa; Mr. C. L. E. Wolfe, University of California. Two applications for membership in the Society were received.

It was decided to hold the summer meeting of the Society at the University of Michigan, on September 2–4. Professors Beman, Bliss, Karpinski, Osgood, and the Secretary were appointed a committee on arrangements for this meeting A committee, consisting of Professors Bliss, Birkhoff, and Veblen, was appointed to prepare nominations for officers and other members of the Council to be elected at the annual meeting in December.

Professors E. W. Brown, L. E. Dickson, and H. S. White were appointed as representatives of the Society in the Division of Physical Sciences of the National Research Council, and President R. S. Woodward and Professors G. D. Birkhoff and W. D. MacMillan as representatives of the Society in the American Section of the International Astronomical Union.

Professor Archibald, chairman of the committee on the publication of a mathematical year book, presented a preliminary report; the committee was continued and asked to make a further report at a future meeting.

Professor A. B. Coble was appointed to succeed Professor D. R. Curtiss as a member of the Editorial Committee of the *Transactions*.

A special feature of the meeting was the reports on the work in ballistics at Aberdeen and Washington, which occupied the first part of the afternoon session. Titles and abstracts of these reports are included in the list below (papers 14, 15, 16).

About fifty members and friends gathered at the midday luncheon. Thirty-two attended the dinner in the evening. Much satisfaction was expressed at the return of these pleasant occasions.

The following papers were read at this meeting:

(1) Professor C. J. KEYSER: "Concerning groups of dyadic relations in an arbitrary field."

(2) Mr. J. K. WHITTEMORE: "Certain functional equations connected with minimal surfaces."

(3) Professor W. B. FITE: "Linear functional differential equations."

(4) Mr. L. B. ROBINSON: "Note on a theorem due to Wilczynski."

(5) Mr. L. B. ROBINSON: "A curious system of polynomials, continued."

(6) Professor O. E. GLENN: "Covariants of binary modular groups."

(7) Professor O. E. GLENN: "Modular covariant theory of the binary quartic. Tables" (preliminary report).

(8) Professor O. E. GLENN: "Invariants of velocity and of acceleration."

(9) Professor F. H. SAFFORD: "Reduction of the elliptic element to the Weierstrass form."

(10) Mr. PHILIP FRANKLIN: "Computation of the complex roots of the function $P(z)$."

(11) Dr. A. R. SCHWEITZER: "On the history of functional equations" (preliminary report).

(12) Professor E. D. ROE, JR.: "The irreducible factors of $1 + x + \cdots + x^{n-1}$. Second paper."

(13) Professor E. D. ROE, JR.: "The irreducible factors of a circulant."

(14) Captain DUNHAM JACKSON: "Contributions of modern mathematics to exterior ballistics."

(15) Dr. T. H. GRONWALL: "Qualitative properties of the ballistic trajectory."

(16) Major OSWALD VEBLEN: "Progress in design of artillery projectiles."

(17) Professor G. D. BIRKHOFF: "Boundary value and expansion problem for differential systems of the first order."

(18) Professor G. D. BIRKHOFF: "Note on the closed curves described by a particle moving on a surface in a gravitational field."

(19) Professor G. D. BIRKHOFF: "Note on the problem of three bodies."

(20) Professor EDWARD KASNER: "A characteristic property of central forces."

(21) Dr. J. F. RITT: "On weighting-factor curves for low elevations."

(22) Professor A. C. LUNN: "Some functional equations in the theory of relativity."

(23) Dr. J. R. KLINE: "Concerning sense on closed curves in non-metrical plane analysis situs."

(24) Professor R. L. MOORE: "On the most general class

L of Fréchet in which the Heine-Borel-Lebesgue theorem holds true."

(25) Mr. H. S. VANDIVER: "On the class number of the field $\Omega(e^{2i\pi/p^n})$ and the second case of Fermat's last theorem."

(26) Professor F. W. BEAL: "On certain points of congruences of circles."

(27) Professor L. L. SILVERMAN: "Regular transformations of divergent series and integrals."

(28) Mr. T. C. FRY: "The application of the modern theories of integration to the solution of differential equations."

(29) Dr. C. A. FISCHER: "Completely continuous transformations and Stieltjes integral equations."

(30) Professor ARNOLD EMCH: "On closed curves described by a spherical pendulum."

(31) Professor H. S. WHITE: "An explicit formula for two old problems."

(32) Professor L. P. EISENHART: "Triply conjugate systems with equal point invariants."

Mr. Franklin was introduced by Dr. Gronwall. The papers of Mr. Robinson, Professor Safford, Dr. Schweitzer, Professor Lunn, Professor Beal, Professor Silverman, Mr. Fry, Dr. Fischer, Professor Emch, Professor White, and Professor Eisenhart were read by title. Abstracts of the papers follow below.

1. If the domain of a relation be identical with its codomain and therefore with its field, then, whatever field be given, any relation of the kind in question belongs to one and but one of the following nine types of relations: one-one, one-some, some-one, some-some, one-many, many-one, some-many, many-some, many-many. These classes together with those formed from them by logical addition constitute a certain class C of classes of relations. Professor Keyser has determined what members of C are groups, the rule of combination being relative multiplication.

2. In Mr. Whittemore's paper functional equations for the Enneper-Weierstrass function defining a real minimal surface, which express certain properties of the surface, are solved, and from the results several theorems are obtained.

3. Professor Fite establishes the existence of solutions of a

certain class of linear functional differential equations and discusses the properties of the functions defined by certain mixed difference equations.

4. At a previous meeting of the Society (April 28, 1917) Mr. Robinson showed that the general system of linear homogeneous differential equations could be reduced to a canonical form whose covariants could be computed by a finite number of quadratures. This result was accepted for publication by the editors of the *Johns Hopkins Circular*. Subsequently the author discovered that his method yields a very neat proof of the theorem enunciated by Wilczynski on page 39 of his Projective Differential Geometry.

5. In this paper Mr. Robinson continues a note read at the summer meeting of 1918. Given a system of equations

$$\sum_{j=1}^{n+1} P_{jj}x_{jr}^2 + 2\sum_{i=1}^{n+1}\sum_{k=1}^{n+1} P_{ki}x_{kr}x_{ir} = 0$$

$$(r = 1, 2, \cdots, n), \qquad (i > k);$$

polarize each of the above equations with respect to the r following points: $x_{1r}, x_{2r}, \cdots, x_{n+1\,r}$ $(r = 1, 2, \cdots, n)$. Denote the resulting system by J_2. The system composed of J_1 and J_2 can be designated by P. Solve J_1 with respect to x_{1r}, substitute the solutions in J_2, and rationalize. Write the coefficients of the resulting forms thus:

$$C_\lambda(P_{11} \cdots P_{n+1\,n+1}).$$

The necessary and sufficient condition for the existence of values of the x's annulling P and not annulling all the determinants of the matrix $\| x_{ij} \|$ $(i = 1, 2, \cdots, n+1)(j = 1, 2, \cdots, n)$ is given by the vanishing of all the C's.

6. Contributions relating to processes, general and enumerative, for complete systems of covariants of binary forms transformed modulo 2, or 3, are treated in Professor Glenn's paper, which is a summary of the tabulated results, and methods, of a number of papers by the author on formal modular covariant theory (mod p).* An algorism of modular

* For a treatment of universal covariants of modular groups, see Dickson, *Transactions*, vol. 12 (1911), p. 75.

convolution, involving simultaneous invariants and covariants, is here given as a new conclusion, and the forms from the complete systems of former papers are exhibited under the notation of this algorism.

The paper appeared in the April number of the *Proceedings of the National Academy of Sciences.*

7. The doctrine of the covariant scale and of the construction of a complete formal modular concomitant system by a process of passing from degree to degree, developed by Professor Glenn in former papers, now enables him to derive fundamental systems of the binary quartic form with reference to the linear congruence groups G_6 (mod 2), and G_{48} (mod 3). He also extends the general theory of the forms of even order transformed modulis 2 and 3, and summarizes certain conclusions in this theory in covariant tables for the forms of orders 4, 6, 8. The problem of the complete seminvariant system is, in the methods of this paper, a phase of a theorem on the reducibility modulo p of the general quartic of order $> p^2 - 1$.

8. In relativity theory the velocity $(\dot{x}, \dot{y}, \dot{z})$ of a point moving uniformly, as observed from a system of reference S, is connected with the apparent velocity $(\dot{x}', \dot{y}', \dot{z}')$ for an observer situated upon a system S', moving relatively to S, by a set of equations of substitution upon the variables $\dot{x}, \dot{y}, \dot{z}; \dot{x}', \dot{y}', \dot{z}'$. Professor Glenn constructs systems of concomitants, both universal and quantitative, in each of two domains, for these transformations. A similar invariant theory of acceleration is developed.

9. A formula for the reduction of the elliptic element to the Weierstrass form and quoted by Biermann as derived from Weierstrass' lectures has been discussed briefly by Greenhill, Enneper and others. Haentzschel has referred to it as the most general solution of the differential equation satisfied by all elliptic functions, and then used it as a basis of extension and criticism of Wangerin's surfaces in a potential problem. Professor Safford has published several articles concerning Wangerin's and Haentzschel's results, and in this paper gives new facts about the formula, showing in particular that it is not a more general form but arises solely from a change of constants.

10. By a method based on an existence proof recently given by T. H. Gronwall, the numerical values of the four complex roots of the P-function associated with the gamma function have been calculated by Mr. Franklin. The values found are: $-1.7262976 \pm 1.2380921i$ and $-3.7264730 \pm .5406746i$.

11. Dr. Schweitzer compiles a first section of a bibliography (with critical notes) of functional equations, excluding references primarily to difference, differential and integral equations, or mixed types of such equations. The present work, which the author hopes to amplify, is divided into I. Classification; II. Range of application; III. Methods of solution. Subdivisions under I are I_1. Simple, and I_2. Composite genesis; I_3. Analytic, and I_4. Non-analytic reference, and under I_4 the further divisions I_{41}. General theories of composition, with the sections I_{411}. Equations in partial functions, and I_{412}. Equations in iterative compositions. The operations considered are mainly distribution, inversion, elimination, iteration, transformation as applied to variables or functions. Attention is called to the relations given by Hankel* and Stolz-Gmeiner† when interpreted as functional equations (cf. Hankel: "Calculus of Operations") and their generalization, and the interesting but apparently little known functional equations discussed by E. Schroeder.‡ In the *Mathematische Annalen*, volume 12, page 483, Schroeder gives an attempted, but erroneous, solution of a distributive equation in the domain of substitution groups.

12. In this paper, which is a continuation of a previous one, Professor Roe derives relations among the roots of $1 - x^n = 0$, various formulas for factorization, and the irreducible factors of $f(x, n) \equiv 1 + x + \cdots + x^{n-1} \equiv 1, n$, of $f(x^m, n) \equiv m, n$ and of $X_n(x^m) \equiv {}^m_n$. $X_n(x) = 0$ gives the primitive nth roots of unity, and $X_n(x) \equiv {}^1_n$ is irreducible. The paper establishes that an infinite number of identities each of two types comprises all the interfunctional expressibility relations among the f's and X's, namely:

$$S_i \equiv F(T_1, T_2, \cdots, T_e),$$

* Theorie des complexen Zahlensysteme (1867), pp. 26–28, 34, 13, 63, 106 and elsewhere.

† Theoretische Arithmetik (1902), chapter III, especially, pp. 52, 53, 38, note 1.

‡ For example, *Archiv* (2) vol. 5; *Crelle*, vol. 90; *Math. Annalen*, vols. 10, 29; and the Vorles. u. d. Algebra der Logik, vol. 1 (1890), pp. 617, 630 et seq.

which states that every S can be expressed as an explicit rational function of the T's, and

$$F_1(S_1, S_2, \cdots, S_r) \equiv F_2(T_1, T_2, \cdots, T_s).$$

In both identities S's and T's represent both of them X's, or both f's, or either one f's and the other X's, making eight subtypes in all. Their development and discussion form a large part of the paper, and they are used in solving the main problem.

The most general solutions are:

For irreducible factors of the X's

$$\frac{n_1^{a_1} \cdots n_r^{a_r} m_1^{c_1} \cdots m_j^{c_j}}{m_1^{b_1} \cdots m_k^{b_k}} \equiv \prod_{\Sigma\lambda_i=0}^{\Sigma\lambda_i=\Sigma a_i} n_1^{\lambda_1} \cdots n_r^{\lambda_r} m_1^{\nu_1} \cdots m_k^{\nu_k},$$

$\nu_i = b_i + c_i$, $k \geqq j \geqq 0$, $a_i \geqq \lambda_i \geqq 0$, and the product contains $\Pi_1^r (a_i + 1)$ irreducible factors.

For irreducible factors of the f's

$$n_1^{a_1} \cdots n_r^{a_r} m_1^{c_1} \cdots m_j^{c_j}, m_1^{b_1} \cdots m_k^{b_k} \equiv \prod_{\Sigma\lambda_i=0,\ \Sigma\nu_i=1+\Sigma c_i}^{\Sigma\lambda_i=\Sigma a_i,\ \Sigma\nu_i=\Sigma(b_i+c_i)} n_1^{\lambda_1} \cdots n_r^{\lambda_r} m_1^{\nu_1} \cdots m_k^{\nu_k},$$

$k \geqq j \geqq 0$, $c_{j+l} = 0$ when $l > 0$, $a_i \geqq \lambda_i \geqq 0$, $b_i + c_i \geqq \nu_i \geqq c_i$, a product that contains

$$\prod_1^r (a_i + 1) \left\{ \prod_1^j (b_i + c_i + 1) \prod_{j+1}^k (b_i + 1) - \prod_1^j (c_i + 1) \right\}$$

irreducible factors.

Numerous applications of the theory and methods are given. It is pointed out that a theorem of Lombardi is contained as a very special case in the second of the preceding general solutions, which is the generalization of all problems of the type of Lombardi's. A theorem proved by De Virieu is also generalized: The necessary and sufficient conditions that $f(x) \equiv \Pi_1^k(x^{\alpha_i} - 1)/\Pi_1^r(x^{\beta_i} - 1)$ is integral are: no prime number enters more β's than α's, and its distribution in the individual β's is not higher than in the individual α's. If $r = k = p$, $\alpha_i = m - i + 1$, $\beta_i = i$, and x is replaced by x^b, we get the theorem proved by De Virieu. The paper concludes with lists of irreducible factors and applications of the developed theory.

13. The observations contained in this paper will be incorporated in Professor Roe's paper preceding.

14. Captain Jackson's paper gives a brief summary of recent progress in the treatment of the fundamental problems of exterior ballistics under the guidance of modern methods in mathematical analysis. Particular emphasis is placed on Moulton's development of the method of numerical integration for computing trajectories and differential variations, and on Bliss's method of computing differential variations by the use of an adjoint system of differential equations. It is pointed out further that, while the theory of the motion of a projectile, regarded as a material particle, has reached a relatively satisfactory state, present knowledge of the accompanying gyroscopic and fluid motions is highly incomplete.

15. In this paper Dr. Gronwall considers the differential equations for the motion of a heavy particle in a resisting medium $x''=-Fx'$, $y''=-Fy'-g$, where F is the resistance divided by the velocity. A qualitative study of the trajectories belonging to this system is made under the sole assumption that $F=F(x', y', x, y, t)$ is a *positive* continuous function of its five arguments. Various inequalities involving range, maximum ordinate, time of flight, etc., are derived, all of these inequalities reducing to equalities in vacuum. A summary of further results obtained under stronger assumptions on F is given; these results are of practical importance in the question of deducing simple approximate formulas for the differential corrections in ballistics.

16. The paper by Major Veblen is a report on experimental work carried out at the Aberdeen Proving Ground, which led to improvements in artillery projectiles. Among the questions investigated were the effects of false ogives, of boat-tailed or stream-lined bases, and of devices for modifying the yaw. Considerable increases in range were obtained by these methods, but the most important gain was obtained by a modification of the rotating band by the removal of some excess copper. This resulted, in the case of the 6-inch gun, in an increase in range of about $2\frac{1}{2}$ miles, and in a diminution of dispersion by a factor of 8. Similar results were obtained in several other guns. It is intended to publish an account

of these studies on rotating bands in an article by Major Veblen and Lieutenant Alger in the *Journal of the United States Artillery*.

These investigations were of an empirical nature, and were carried out, for the most part, without the guidance of a mathematical theory. The absence of a mathematical theory is particularly to be underlined in the case of the rotating band experiments, because semi-official popular accounts of this work have been published in which it is stated that the results were obtained by pure mathematics. The work at the proving ground was done in close collaboration by several men, among whom may be mentioned Lieutenants P. L. Alger, F. E. Fish, and H. Galajikian, and Captain J. W. Alexander. Although the coöperation among all concerned was so close as to make it impossible to trace ideas to their sources, the largest share of the credit must be assigned to Lieutenant Alger, particularly in the rotating band investigations. The results on the 6-inch gun were obtained independently in France by Captain R. H. Kent, after studying the Aberdeen results for the 8-inch howitzer.

17. Professor Birkhoff shows that his earlier work on boundary value and expansion problems for a single ordinary linear differential equation of the nth order (*Transactions*, volume 9 (1908), pages 219–231, 373–395) can be extended to differential systems of the first order. In this manner it is possible to deal with these problems for ordinary differential equations of the nth order in which the parameter does not enter linearly.

18. In a paper by Professor Emch (*Proceedings of the National Academy of Sciences*, volume 4 (1918), pages 218–221) the existence of certain types of closed curves of motion for a particle moving on a sphere in a gravitational field is established. Professor Birkhoff points out in his note that if a point moves on any surface which has a vertical plane of symmetry toward which it is concave, then the existence of such closed curves is an immediate consequence of general results deduced by him earlier (*Transactions*, volume 18 (1917), pages 193–300).

19. Using the methods of Sundman (*Acta Mathematica*,

volume 36 (1912), pages 105–195), Professor Birkhoff proves that if three bodies lie sufficiently near together at $t = 0$ with *assigned* area integral constants not all zero, then at least two of the mutual distances become infinite for lim $t = \infty$.

20. Professor Kasner shows that the differential equation of the third order defining all the trajectories of a plane field of force admits a first integral of the form $y'' = cF(x, y, y')$ when and only when the field is central (or, as a limiting case, parallel). Generalizations are obtained by imposing some instead of all five of the geometric properties characterizing a dynamical family. (See Princeton Colloquium Lectures, page 10.)

21. In this paper Dr. Ritt discusses the manner in which weighting-factor curves for differential variations behave as the elevation approaches zero. It is found that there exists a limiting weighting-factor curve for each type of atmospheric disturbance. In the case of a range wind the weighting-factor curve is a cubic which depends on the muzzle velocity, but not on the projectile. The weighting-factor curves for a variation in air density and for a variation in the velocity of sound have the same weighting-factor curve for zero elevation, a cubic which is the same for all projectiles and for all velocities. The weighting-factor curves for cross wind approach a parabola which is also independent of the projectile and of the velocity. These limiting curves may be used in the reduction of firings at low angles, when it is desired to avoid short-arc computations. This paper will be offered for publication to the *Journal of the U. S. Artillery.*

22. In transcribing his process of light-signalling Einstein obtained relations connecting the space-time coordinates in two systems, in the form of functional equations. He solved these by passing to partial differential equations and restricted the solutions to linearity by appeal to a homogeneity whose meaning is somewhat obscure. In the present paper, through a somewhat more complete use of functional equations, Professor Lunn shows that the assumption of differentiability is not necessary, and that the needed features of homogeneity are already implicit in the optical and kinematic postulates.

23. In a paper, recently published in the *Annals*,* Dr. Kline gave a non-intuitional definition Σ of sense on closed curves in any space satisfying Professor R. L. Moore's system of axioms Σ_3. It will be remembered that Σ_3, besides being satisfied by an ordinary euclidean plane, is also satisfied by certain spaces which are neither metrical, descriptive, nor separable. In the present paper, Dr. Kline considers the following set of independent postulates for sense on closed curves with respect to their exterior:

Axiom 1. If A and C separate B and D on the closed curve J, then the sense ABC on J is not the same as the sense ADC on J.

Axiom 2. If, on the arc ABC of the closed curve J, E, H and F are points such that on the arc ABC the order $AEHFC$ holds, while EXF is an arc lying except for its endpoints within J, then the sense ABC on J is the same as the sense EHF on $EHFXE$.

Axiom 3. If the sense $A_1B_1C_1$ on J_1 is both the same as the sense $A_2B_2C_2$ on J_2 and as the sense $A_3B_3C_3$ on J_3, then the sense $A_2B_2C_2$ on J_2 is the same as the sense $A_3B_3C_3$ on J_3.

The result obtained is that, if Σ' is any definition of sense on closed curves in a space satisfying Σ_3, then a necessary and sufficient condition that Σ' have the properties of Axioms 1 to 3 is that Σ' be equivalent to Σ.

24. In a class L of Fréchet, a set of points M is said to possess the Heine-Borel-Lebesgue property if, for every infinite family G of subsets of L that covers M, there exists a finite sub-family of G that also covers M. The Heine-Borel-Lebesgue theorem is said to hold true in a given space L if every closed and compact subset of L possesses the Heine-Borel-Lebesgue property. In a recent paper, Fréchet has shown that in order that the Heine-Borel-Lebesgue theorem should hold true in a given class S (i.e., a class L in which the derived set of every set is closed) it is sufficient that that class S should be a class V (a class L in which a distance function exists). He points out however that this sufficient condition is not necessary and raises the question as to what property it is necessary and sufficient that a class S should possess in order that the Heine-Borel-Lebesgue theorem should hold

* Cf. "A definition of sense on closed curves in non-metrical plane analysis situs," *Annals of Mathematics*, 2d series, vol. 19 (1918), pp. 185–200.

true in that class. In the present paper, Professor Moore exhibits one such property. He calls a family G of point sets a monotonic family if, of every two members of G, one of them contains the other one. A monotonic family G is said to be proper if there is no point common to all the members of G. A set of points M is said to have the property K if every monotonic family of closed subsets of M is improper. A necessary and sufficient condition that the Heine-Borel-Lebesgue theorem should hold true in a given class S is that in that class every closed compact point set should have the property K.

If a set of points M is defined as being compact in the new sense in case for every proper monotonic family G of subsets of M there exists at least one point which is a limiting point of every member of G, then the following theorem holds true:

In a class S, in order that a point set M should possess the Heine-Borel-Lebesgue property it is necessary and sufficient that M should be closed and compact in the new sense.

25. In the *Göttinger Nachrichten*, 1910, pages 507–516, F. Bernstein proves that if the class number of the field defined by $e^{2i\pi/p^2}$, where p is prime, is divisible by p but not by p^2, then the relation

$$(px)^p + y^p + z^p = 0$$

is not satisfied in rational integers x, y, z, none zero. In the present paper Mr. Vandiver investigates the divisibility of $\Omega(e^{2i\pi/p^2})$ by p, and shows that Bernstein's assumption is equivalent to the statement that p is a regular prime, and therefore his result is included in Kummer's well known criterion.

26. Professor Beal's paper is in abstract as follows: With one exception, at most, four points of a circle of a congruence generate surfaces normal to lines through them and in planes orthogonal to the circle. If the circle touches the envelope S of its plane, one, two, or three points may coincide at the point of tangency without imposing a condition on the radius R. If S is developable or if the circle is tangent to an asymptotic line, two coincide.

When the center is at the point of tangency these cases arise: With radius $R(u)$ and lines of curvature parametric,

two points are on the tangent to $v = \text{const.}$ and the normals at them intersect the normal of S at distances $\pm R'(u)\mathcal{E}^{-1/2}$ from the center of normal curvature of $v = \text{const.}$* The other points are symmetrical with respect to the tangent of $v = \text{const.}$ and the normals pass through the center of normal curvature of $u = \text{const.}$ The last two points are real only when GR'^2 is less than or equal to $\mathcal{E}G + GE + 2DD''$. In the first case the points are distinct and in the second coincide on the tangent to $v = \text{const.}$ The above holds true if R is constant. The four points are then on the tangents to the lines of curvature. If S is developable with rectilinear generators $v = \text{const.}$ and orthogonal trajectories $u = \text{const.}$ three cases exist: $(\partial R/\partial u)^2 < E$, four distinct points; $(\partial R/\partial u)^2 = E$, three coincident points; $(\partial R/\partial u)^2 > E$, two imaginary points.

27. Hurwitz and Silverman have studied transformations of sequences of the type $t_n = \sum_{k=1}^{n} a_{nk}s_k$, where the numbers a_{nk} are defined in terms of certain values of a function $f(z)$ of the complex variable z. Professor Silverman defines a_{nk} by means of a real function $f(x)$ of the real variable x. The condition for regularity of the transformation T_f is obtained simply in terms of $f(x)$ and its continuous derivative $f'(x)$. The condition for the equivalence of two transformations T_f and T_g is that there exist a solution, having certain properties, of an integral equation involving $f(x)$ and $g(x)$. As a special case, the equivalence of Cesàro and Hölder summability of like order, integral or fractional, is deduced. It is further shown that this method in terms of real variables is more general than the one in terms of complex variables.

The results obtained for divergent series are shown to apply also to divergent integrals.

28. The primary object of Mr. Fry's paper is to establish the validity of certain methods of treating differential equations by the use of Fourier's integrals. These methods have been used by a number of mathematicians in the treatment of circuit problems, but the very considerable extent of their field of validity apparently has not been recognized. From

*E, G, D, D', $\mathcal{E}$ and G are fundamental quantities for S and its spherical representation.

a standpoint of pure mathematics, the most important contribution of the paper is a definition of the integral

$$\int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} dn f(\lambda)\phi(n, t)e^{in(t-\lambda)},$$

which applies even when ϕ does not vanish at ∞. This definition is shown to be consistent with the ordinary Riemann definition, when the latter applies; and the validity of performing addition, subtraction, differentiation and integration under the sign of integration is established. The conception of Stieltjes' integrals is used in carrying out the argument. A minor point of interest concerning the possibility of varying the path of a Fourier integral is touched upon, leading to the reduction of such integrals to closed paths.

29. Riesz has recently derived a large part of the Fredholm theory of integral equations for the equation

$$f(x) = \varphi(x) + \lambda T(\varphi),$$

where $T(\varphi)$ is any completely continuous linear transformation, and in an earlier paper he has proved that every linear transformation, that is linear functional depending on a parameter, can be put in the form

$$T(\varphi) = \int_a^b \varphi(y) d_y K(x, y).$$

After deriving a sufficient condition that a set of discontinuous functions shall be compact, Dr. Fischer has proved that this transformation will be completely continuous if the variation of K in y and its two-dimensional variation are both finite. Another sufficient condition and a necessary condition are also obtained.

30. Professor Emch presents the results of an investigation of the curves described by a spherical pendulum in case that they are closed. The discussion is based upon the parametric representation of the cartesian coordinates of a point of the curve by Weierstrassian σ-functions. It is shown that the closed curves are algebraic, and rational in Greenhill's case, in which the pendulum-bob just reaches the plane of suspension. A classification of these curves is given as well as an

account of their principal geometric properties. The paper was published in the March number of the *Tôhoku Mathematical Journal.*

31. In the theory of conics it is a well known theorem that two inscribed triangles determine a second conic, for which each is self-polar. On a twisted cubic curve a similar theorem was proved by von Staudt and Hurwitz for two tetrahedra. Professor White's paper presents an explicit algebraic formula by which both these theorems are established, and from whose obvious extension others can be inferred. This formula effects in the one case a (2, 2) transformation, in the other a (3, 3) transformation, of the parameter in terms of which points on the locus are rationally expressible. It permits the explicit determination of the infinitely many self-polar triangles of a second conic which are inscribed in the first, as soon as two are known.

32. When the coordinates of a point in space are expressed in terms of three independent parameters u, v, w, the points for which any one of the parameters is constant, say w, lie on a surface. If the parameters are such that the curves $u = \text{const.}$ and $v = \text{const.}$ on the surface form a conjugate system, we say that space is referred to a triply conjugate system of surfaces. Professor Eisenhart proposes and solves the problem of finding triply conjugate systems for which the three Laplace equations satisfied by the cartesian coordinates have equal invariants. When in particular the triple system of surfaces is orthogonal, the lines of curvature on all the surfaces are isothermic. The problem of finding these orthogonal systems has been solved by Darboux. This paper will appear in the June *Annals.*

F. N. Cole,
Secretary.

REPORT ON THE THEORY OF THE GEOMETRY OF NUMBERS.*

BY PROFESSOR H. F. BLICHFELDT.

(Read at the Chicago Symposium of the American Mathematical Society, March 28, 1919.)

1. *Introduction.*—There arises in theory of numbers an important class of problems of a kind best illustrated by an example: Consider the quadratic form $f = ax^2 + 2bxy + cy^2$, $ac - b^2 > 0$. Let the numerical value of the determinant $D = ac - b^2$ be given, but not the coefficients a, b, c individually; also let it be specified that the variables x, y must be integers (positive or negative or zero). Under these conditions, what can be predicted as to the least absolute value of f, other than $f = 0$? Designating this value by $[f]$, we have† $[f] \leqq 2/\sqrt{3D^1}$, the extreme limit being reached by the form $x^2 + xy + y^2$.

Isolated problems of like nature were studied by prominent mathematicians during the past century. Hermite discovered a superior limit to the least value $[f]$ of a positive definite quadratic form in n variables in terms of n and the numerical value of the determinant D: $[f] \leqq (\frac{4}{3})^{(n-1)/2} D^{1/n}$ (*Journal für Mathematik*, volume 40, 1850, page 263); this was the first important result of a general nature.

In the matter of references we shall use the abbreviations:

M_1, M_2, M_3, M_4 designate respectively the following books by Minkowski: "Diophantische Approximationen," Leipzig, 1907; "Geometrie der Zahlen," Leipzig, 1896–1910; "Gesammelte Abhandlungen," volumes 1 and 2, Leipzig, 1911.

B_1 refers to "A new principle in the geometry of numbers" by the author, *Transactions American Mathematical Society*, 1914, pages 227–235; B_2 to a paper read before this Society, San Francisco Section, April 6, 1918 (see this BULLETIN, 1918, page 418).

* An exposition of the theory of the geometry of numbers is to appear in the *Annals of Mathematics* during the fall and winter of the present year. For this reason the report here given of the lecture at the Chicago Symposium is very brief. Proofs have been omitted, and only a few illustrative examples are included. The fundamental theorems are, however, stated practically in full (§3).

† See B_1, p. 233, for references.

2. *Minkowski's "nowhere concave" surfaces* (M$_2$, pages 1–76).—It remained for Minkowski to discover a theorem bearing on the least values of a very general class of functions, by means of an elegant geometrical interpretation of this minimum. He defines a real function φ of n real finite variables $x_1, \cdots, x_n$ having the following properties (A), (B), (C):

(A) $\varphi(x_1, \cdots, x_n) =$ a definite positive number unless $x_1 = 0, \cdots, x_n = 0$;

$\varphi(0, \cdots, 0) = 0$;

$\varphi(tx_1, \cdots, tx_n) = t\varphi(x_1, \cdots, x_n)$ for any positive number t;

(B) $\varphi(x_1 + y_1, \cdots, x_n + y_n) \leqq \varphi(x_1, \cdots, x_n) + \varphi(y_1, \cdots, y_n)$;

(C) $\varphi(-x_1, \cdots, -x_n) = \varphi(x_1, \ldots, x_n)$.

The following functions (1) and (2) may serve as illustrations. We have taken $n = 2$, $x_1 = x$, $x_2 = y$.

(1) $\varphi = \sqrt{ax^2 + 2bxy + cy^2}$, $ac - b^2 > 0$;

(2) $\varphi = |x|$ when $|x| \geqq |y|$; $\varphi = |y|$ when $|x| \leqq |y|$.

The corresponding curves $\varphi = k$ are respectively an ellipse and a square.

It may be proved that

(*a*) φ is continuous in the variables $x_1, \cdots, x_n$;

(*b*) the variables satisfying the inequality $\varphi \leqq k$, for a given finite positive k, do not exceed in absolute value a certain finite number depending on k and φ;

(*c*) the surface $\varphi = k$ has the origin $(0, \ldots, 0)$ as a center of symmetry and is nowhere convex as seen from this center;

(*d*) for a given positive k, the points $(x_1, \ldots, x_n)$ are separated into three distinct sets: inner points, points on the surface, outer points, according as the following conditions are satisfied: $\varphi < k$, $\varphi = k$, $\varphi > k$;

(*e*) the outer and inner volumes of $\varphi = k$ have one and the same value, namely the integral $\int dx_1 dx_2 \cdots dx_n$ extended to all the inner points.

It follows that only a finite number of lattice points, (i. e., points whose coordinates are integers (positive or negative or zero), will be inner points or points on the surface $\varphi = k$,

and that the surface will lie in the interior of a cube (n-dimensional) of finite edge. If k is taken small enough, all lattice points other than the origin will be outer points.

3. *General theorems.* For a surface as thus defined, Minkowski proved the following theorems I, I′, III:

THEOREM I (M_2, page 76). *If the volume of $\varphi = k$ is $\geqq 2^n$, then at least one lattice point other than $(0, \cdots, 0)$ will be an inner point or a point on the surface.*

THEOREM I′ (M_2, page 76). *If the volume of $\varphi = 1$ is represented by J, then there will be at least one set of integers $l_1, \cdots, l_n$ not all zero, such that*

$$0 < \varphi(l_1, \cdots, l_n) \leqq 2/J^{1/n}.$$

The somewhat more general geometrical theorem proved by the author (B_1, page 228) may be stated as follows in condensed form:

THEOREM II. *A simple closed surface of volume V can be placed in such a position by means of a translation $x_1 = x_1' + a_1, \cdots, x_n = x_n' + a_n$ that the number of lattice points which will lie inside or on the surface is $> V$.*

This surface need not possess a center, not does it need to be concave towards an inner point.

THEOREM III (M_3, page 270). *There exists a function φ' obtained from φ by means of a linear homogeneous transformation of determinant unity of the variables $x_1, \cdots, x_n$, such that*

$$[\varphi'] \geqq \left\{ \frac{2\left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots\right)}{J} \right\}^{1/n}$$

Geometrically: if the volume of $\varphi' = k$ is

$$< 2\left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots\right),$$

then will every lattice point except $(0, \cdots, 0)$ be an outer point. The author has obtained slightly better results (B_2).

4. *Applications to homogeneous forms.*

THEOREM IV (M_1, pages 68–79). *Let $\xi = \lambda x + \lambda' y + \lambda'' z$, $\eta = \mu x + \cdots$, $\zeta = \nu x + \cdots$ be three linear forms in x, y, z having real coefficients $\lambda, \cdots, \nu''$ and a determinant $\Delta = (\lambda\mu'\nu'') \neq 0$. Then*

(a) *integers* l_1, m_1, n_1, *not all zero, may be substituted for* x, y, z, *—say* $\xi_1 = \lambda l_1 + \lambda' m_1 + \lambda'' n_1, \cdots$,*—such that*

$$|\xi_1| < \sqrt[3]{|\Delta|}, |\eta_1| < \sqrt[3]{|\Delta|}, \quad |\zeta_1| \leqq \sqrt[3]{|\Delta|};$$

(b) *integers* l_2, m_2, n_2, *not all zero, may be substituted for* x, y, z *such that*

$$|\xi_2| + |\eta_2| + |\zeta_2| < \sqrt[3]{6|\Delta|}.$$

These results follow immediately from Theorem I′, the corresponding surface $\varphi = 1$ being a parallelopiped and an octahedron respectively. By a special analysis of the octahedron Minkowski proved (M4, page 40) that the left-hand member under (b) is $\leqq \sqrt[3]{\frac{108}{19}}|\Delta|$.

From this result we derive $|\xi_2\eta_2\zeta_2| \leqq \frac{4}{19}|\Delta|$.

Similar results are obtained when the coefficients of ξ and η are conjugate imaginary ($\eta = \bar{\xi}$) and those of ζ real. Thus we find (B1, page 233, from Theorem III)

$$|\xi_3\bar{\xi}_3\zeta_3| < \sqrt{\tfrac{2}{7}}|\Delta|.$$

THEOREM V. *Let* f *be a positive definite quadratic form in* n *variables and of determinant* D. *Then integers, not all zero, may be substituted for the variables such that the numerical value of* f *is*

$$[f] \leqq \frac{4}{\pi}\left\{\Gamma\left(1+\frac{n}{2}\right)\right\}^{2/n} D^{1/n}.$$

The author has obtained the limit

$$\frac{2}{\pi}\left\{\Gamma\left(1+\frac{n+2}{2}\right)\right\}^{2/n} D^{1/n}$$

by means of Theorem II (B1, page 233). The *exact* (superior) limit to the lowest value $[f]$ (i. e., the limit actually reached by at least one quadratic form of determinant D) has been determined for $n = 2, 3, 4, 5$ (see B1, page 233). For any n, we know from Theorem III above that this exact limit, which we shall denote by $4(\rho/J)^{2/n}$, cannot fall below

$$\frac{1}{\pi}\left\{2\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\cdots\right)\Gamma\left(1+\frac{n}{2}\right)\right\}^{2/n} D^{1/n};$$

i. e.,

$$\frac{1+1/2^n+1/3^n+\cdots}{2^{n-1}} \leqq \rho \leqq \frac{n/2+1}{2^{n/2}}. \tag{1}$$

Hence, quadratic forms exist for which the corresponding values of ρ satisfy these inequalities. However, no such form has ever been constructed for large values of n.

The problem of the closest *regular* (latticed) packing of spheres in R_n is equivalent to the problem of finding that positive definite quadratic form in n variables and of given determinant, say $D = 1$, whose least value, other than zero, is the highest possible for the given n and D (M4, page 74 ff.). The ratio of the space occupied by spheres packed in regular layers in a large cube, say, to the volume of the whole cube, is indeed the number ρ defined above.

The author has recently proved, by an essentially different method, that no matter how the spheres be packed in a large volume V, in a "regular" fashion or not, the ratio of the space occupied by the spheres to the whole volume V is

$$< \frac{n/2 + 1}{2^{n/2}}.$$

It may be of interest to note in passing that it follows from the inequalities (1) satisfied by ρ for certain (though unknown) quadratic forms, that the shot-pile packing of spheres, though the closest packing in space of two dimensions and presumably also in space of three dimensions, is very far from being the closest packing in space of a large number of dimensions (M4, page 95).

APPLICATIONS OF THE GEOMETRY OF NUMBERS TO ALGEBRAIC NUMBERS.

BY PROFESSOR L. E. DICKSON.

(Read at the Chicago Symposium of the American Mathematical Society, March 28, 1919.)

1. THE geometry of numbers not only furnishes a concrete geometric image of certain fundamental theorems on algebraic numbers, but also provides a new and attractive method of proving important theorems on algebraic fields. For the sake of concreteness we shall restrict attention to the typical case of the cubic field $F(\theta)$, which is composed of the numbers

$r = X + Y\theta + Z\theta^2$, where X, Y, Z are rational numbers, and θ is a root of a cubic equation with integral coefficients, that of θ^2 being unity. A root of any such equation is called an integral algebraic number.

2. *Basis of a Field.*—While the above number r is an integral algebraic number if X, Y, Z are integers, the converse need not be true. However, there exist integral algebraic numbers ω_1, ω_2, ω_3 of the field $F(\theta)$ such that the linear functions of ω_1, ω_2, ω_3 with integral coefficients give all the integral algebraic numbers of the field. The essential idea of the geometric proof consists in showing that the points (X, Y, Z) which correspond to the integral algebraic numbers r of our field form a lattice determined by the fundamental parallelopiped whose edges OA, OB, OC are defined as follows: Of the points on the positive X-axis which correspond to integral algebraic numbers of $F(\theta)$, let A be the one nearest the origin O and distinct from O. Of the points in the XY-plane having $Y > 0$ and corresponding to integral algebraic numbers of $F(\theta)$, let B be a point lying as near the X-axis as possible. Finally, of the points with $Z > 0$ and corresponding to integral algebraic numbers of our field, let C be one as near as possible to the XY-plane.

3. *Basis of an Ideal.*—A set of integral algebraic numbers of a field is called an ideal if the sum and difference of any two equal or distinct numbers of the set are numbers of the set and if the product of any number of the set by any integral algebraic number of the field equals a number of the set. Consider the lattice whose points correspond to the integral algebraic numbers $x\omega_1 + y\omega_2 + z\omega_3$ of our field, x, y, z being integers. The points of this lattice which correspond to the numbers of any ideal form a new lattice with a fundamental parallelopiped determined by OA_1, OA_2, OA_3. If α_i is the number corresponding to the point A_i, the numbers of the ideal are the linear combinations, with integral coefficients, of α_1, α_2, α_3, which therefore form a basis of the ideal.

4. *Units of a Field.*—By a unit ϵ is meant an integral algebraic number which divides unity, so that $1/\epsilon$ equals an integral algebraic number. Dirichlet* stated, but did not publish a detailed proof of, the following fundamental theorem: If among the fields conjugate to a field F of degree m, r are real and the remaining $m - 2r = 2c$ form c pairs of conjugate

* *Sitzungsber. Akad. Wiss. Berlin*, 1846, pp. 103–7; Werke, I, p. 639 (cf. p. 622, p. 630).

imaginary fields, there exist $s = r + c - 1$ units $\epsilon_1, \ldots, \epsilon_s$ in F such that every unit ϵ of F can be expressed in one and but one way in the form

$$\epsilon = \rho \epsilon_1^{a_1} \cdots \epsilon_s^{a_s},$$

where $a_1, \ldots, a_s$ are rational integers, positive, negative or zero, while ρ is a root of unity belonging to the field F. Kronecker* gave a proof along the lines suggested by Dirichlet. Another algebraic proof had been given by Dedekind.†

Minkowski‡ gave a new proof which is largely algebraic, but makes some use of his geometric results. Later, he gave§ a more purely geometric form to his proof, confining his discussion to the typical cases of cubic and quartic fields. This proof makes use of various ideas and results of the geometry of numbers and shows the power and attractiveness of the latter subject.

5. *Classes of Ideals.*—In his proof of the finiteness of the number of classes of ideals in a cubic field, Minkowski|| made use of his results on the upper bound of a product of three linear forms. The process enables us actually to find representatives of the various existing classes of ideals. This problem would be simplified by the discovery of more exact upper bounds.

PRODUCTS OF SKEW-SYMMETRIC MATRICES.

BY PROFESSOR A. A. BENNETT.

IN the March (1919) number of the BULLETIN occurs (page 281) the following: The philosophical faculty of the University of Berlin announces the following prize problem: "To determine by means of the theory of elementary divisors, the criteria that a given matrix be capable of representation as the composition of two skew-symmetric matrices." A dis-

* *Comptes Rendus*, Paris, vol. 96, 1883; vol. 99, 1884; Werke, III_1, pp. 1–30.

† Dirichlet-Dedekind, Zahlentheorie, ed. 2, 1871, § 166, pp. 471–9; ed. 3, 1879, § 177, pp. 555–567; ed. 4, 1894, § 183, pp. 590–603. Cf. Hilbert, *Jahresbericht der Deutschen Math.-Vereinigung*, vol. 4, 1894–5, pp. 214–222.

‡ Geometrie der Zahlen, 1896, pp. 135–147.

§ Diophantische Approximationen, 1907, pp. 133–148.

|| Diophantische Approximationen, pp. 162–167.

cussion of this topic appears therefore of timely interest. To avoid prolixity, only the case of non-singular matrices will be here considered.

For the sake of brevity and clarity the order of the proof will not be interrupted to prove elementary lemmas which in themselves present no difficulty, but several of such lemmas will be stated without proof at the start. Some of these are proved in all discussions of elementary divisors and others while not so familiar present no new difficulties. The definitions and notations employed are those of Bôcher: Introduction to Higher Algebra.

Some Preliminary Lemmas and Definitions.

1. If P is any non-singular matrix, there exist uniquely defined non-singular matrices P' and P^{-1}, known respectively as the conjugate or transposed, and as the inverse of the given matrix. Further, $(P')^{-1} = (P^{-1})'$ and $P^{-1}P = PP^{-1} = I$, where I is the identity, i. e., $MI = IM = M$ for every square matrix M of the same order as I.

2. Skew-symmetric matrices of odd order are necessarily singular, but non-singular skew-symmetric matrices exist of all even orders.

3. If A and B be square matrices of the same order m and with constant elements, the matrix $A + \lambda B$, where λ is variable, is called a linear λ-matrix. The greatest common divisor of all determinants of order j, formed by suppressing $n - j$ rows and $n - j$ columns of $A + \lambda B$, will be a polynomial in λ. If the arbitrary numerical multiplier in the definition of the greatest common divisor be so taken that the non-vanishing term of highest degree in λ has unity for coefficient, this polynomial is denoted by $D_j(\lambda)$. The ratio $D_i(\lambda)/D_{i-1}(\lambda)$, where $D_0(\lambda)$ is arbitrarily defined as unity, is called the ith invariant factor, and is denoted by $E_i(\lambda)$. For i odd, E_i is called an odd invariant factor, and for i even, an even invariant factor.

4. Operations on matrices which merely interchange rows or columns, or multiply rows or columns by non-vanishing factors, or replace a given row by the sum of this row and another, or likewise with columns, are called elementary transformations.

5. If each of two linear λ-matrices is carried into the other by means of elementary transformations, these have the same

invariant factors. Hence, in particular, $PMP^{-1} - \lambda I$ and $M - \lambda I$ have the same invariant factors, for any matrix P of the same order, since the existence of P^{-1} implies that P is non-singular.

6. There exists a standard skew-symmetric matrix T of order $2n$, defined by the condition that if t_{ij} be a general element of T, $t_{ij} = 0$, for $j \neq 2n - i$, $t_{i,\,2n-i} = -1$, $0 < i \leqq n$, $t_{i,\,2n-i} = +1$, $n < i \leqq 2n$. This standard matrix T is called "standard," for the reason that for any non-singular skew-symmetric matrix S of order $2n$, there is a non-singular matrix, P, such that $PSP' = T$.

7. If a matrix K of order $2n$ be such as to have throughout among its elements $a_{ij} = a_{n+i,\,n+j}$, $i \leqq n$, $j \leqq n$, and $a_{ij} = 0$, if $i \leqq n < j$ or $j \leqq n < i$, then the submatrix K_0 consisting of the elements a_{ij}, where $i \leqq n$, $j \leqq n$, determines the invariant factors of K. Indeed if $F_1, F_2, \ldots, F_n$, be the invariant factors of K_0, $E_{2i-1} = E_{2i} = F_i$, $i = 1, 2, \cdots, n$, where E_j is the jth invariant factor of K, $j = 1, 2, \cdots, 2n$.

8. If S_1 and S_2 are any two non-singular skew-symmetric matrices of the same order, $2n$, the linear λ-matrix $S_1 + \lambda S_2$ may be reduced by elementary transformations to a form $L + \lambda T$, where T is the standard matrix defined above, and the elements a of L satisfy the conditions that $a_{i,\,2n-h} = -a_{2n-h,\,i}$, $i \leqq n$, $h \leqq n$, $a_{i,\,2n-h} = 0$, $i \leqq n \leqq h$, or $h < n < i$. Further specialization of the skew-symmetric matrix L is also always possible.

Discussion of the Problem.

Let M be the given non-singular matrix for which it is desired to determine whether M is of the form S_1S_2, where S_1, S_2 are skew-symmetric matrices. By (2), M must be of even order, say $2n$, since otherwise both S_1 and S_2 would be singular and hence M also. One may select a non-singular matrix, P, such that $PMP^{-1} = (PS_1P')(P'^{-1}S_2P^{-1})$ is of the form $M_1 = TS$. The matrix $M_1 - \lambda I$, or $TS - \lambda I$, becomes, on changing the sign of the first n rows and rearranging rows, $S + \lambda T$, the S and T having the same meaning as previously. Using (8), this may be put in the form $L + \lambda T$. Changing the sign of the last n rows and again rearranging rows, this may be written $K - \lambda I$, K being as in (7). From this one has the following conclusion:

THEOREM. *A necessary and sufficient condition that a non-singular matrix, M, shall be expressible as the product of two skew-symmetric matrices, viz., $M = S_1S_2$, is that every even invariant factor of the linear λ-matrix, $M - \lambda I$, shall be equal to the preceding odd invariant factor.**

UNIVERSITY OF TEXAS.

ON THE FIRST FACTOR OF THE CLASS NUMBER OF A CYCLOTOMIC FIELD.

BY MR. H. S. VANDIVER.

(Read before the American Mathematical Society April 27, 1918.)

LET l be an odd prime rational integer and consider the cyclotomic field defined by $e^{2i\pi/l}$. A number of questions connected with this field depend on the divisibility of its class number by l and its powers. This class number can be expressed as the product of two integral factors one of which (generally referred to as the first factor) is

$$h = \frac{f(Z)f(Z^3)\cdots f(Z^{l-2})}{(2l)^{\frac{1}{2}(l-3)}}, \tag{1}$$

where

$$f(x) = r_0 + r_1x + r_2x^2 + \cdots + r_{l-2}x^{l-2},$$

$Z = e^{2i\pi/l-1}$, r is a primitive root of l, and r_i is the least positive residue of r^i, modulo l.

Kummer† proved that the necessary and sufficient condition that h be divisible by l is that one of the numbers of Bernoulli, B_s, $[s = 1, 2, \cdots, (l-3)/2]$ is divisible by l, a B being termed divisible by an integer i when its denominator is prime to i and its numerator is divisible by i. Kronecker‡ gave another proof which was reproduced by Hilbert.§

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* Otherwise expressed, the condition is that the number of integers within parentheses in the characteristic shall always be even, and these alike in pairs. Thus [(2, 2); (3, 3, 1, 1); (1, 1)] is possible, while [(2, 1); (2, 1)]; [2], and [1, 1] are impossible.

† *Journal für die Mathematik*, vol. 40 (1850.)

‡ *Werke*, vol. 1, p. 93.

§ Die Theorie der algebraischen Zahlkörper, Bericht, p. 429.

In the present paper I give an expression for the residue of h modulo l^n, where n is arbitrary, from which it is possible to give necessary and sufficient conditions that h be divisible by a given power of l in terms of Bernoulli numbers. The argument used is a bit different from those employed by Kummer and Kronecker for the special case $n = 1$.

The decomposition of l into ideal prime factors in the field $\Omega(Z)$ shows that one of these prime factors is

$$\mathfrak{P} = (Z - r, l),$$

whence

$$Z \equiv r (\text{mod } \mathfrak{P})$$

and

$$(2) \qquad Z^{kl^{a'}} \equiv r^{kl^{a'}} \pmod{\mathfrak{P}^{a'+1}}, \qquad a' > \frac{l-3}{2},$$

where k is an integer. The right-hand member of (1) is unaltered by the substitution $(Z^{l^{a'}}/Z)$; hence from (1) and (2)

$$(2l)^{\frac{1}{2}(l-3)}h \equiv \prod_s f(r^{sl^{a'}}) \pmod{\mathfrak{P}^{a'+1}},$$

where $s = 1, 3, \cdots, l-2$. Since h and $f(r)$ are rational integers and l is not divisible by $\mathfrak{P}^2$, we then obtain, if $a = a' - \frac{1}{2}(l-3)$

$$(3) \qquad h \equiv \frac{\prod_s f(r^{sl^{a'}})}{(2l)^{\frac{1}{2}(l-3)}} \pmod{l^{a+1}}.$$

We also have

$$r_i \equiv r^i \pmod{l}, \quad r^{isl^{a'}} \equiv r_i^{sl^{a'}} \pmod{l^{a'+1}}$$

and this gives

$$(3a) \qquad \frac{f(r^{sl^{a'}})}{l} \equiv \sum_{n=1}^{l-1} \frac{n^{sl^{a'}+1}}{l} \pmod{l^{a'}}.$$

But since

$$n^{l^a(l^{a'-a}-1)} \equiv 1 \pmod{l^{a+1}},$$

then

$$\sum_{n=1}^{l-1} \frac{n^{sl^{a'}+1}}{l} \equiv \frac{\sum_{n=1}^{l-1} n^{sl^a+1}}{l} \pmod{l^a},$$

and (3) and (3a) give

$$h \equiv \frac{\prod_{s} \sum_{n=1}^{l-1} n^{sl^{a}+1}}{(2l)^{\frac{1}{2}(l-3)}} \pmod{l^{a}}. \tag{4}$$

We shall now show that, for s odd,

$$\sum_{n=1}^{l-1} n^{sl^{a}+1} \equiv (-1)^{(sl^{a}-1)/2} l B_{(sl^{a}+1)/2} \pmod{l^{a+1}}. \tag{5}$$

By the Bernouilli summation formula we have if $b_1 = -1/2$, $b_{2a+1} = 0$, and $b_{2a} = (-1)^{a+1}B_a$, where $B_1 = 1/6$, $B_2 = 1/30$, etc., $b_s = b^s$,

$$\sum_{n=1}^{l} n^{k} = lb^{k} + l\frac{k}{2} lb^{k-1} + l^{2}\frac{k(k-1)}{2\cdot 3} lb^{k-2} + \cdots + \frac{l^{r}}{r+1}\binom{k}{r} lb^{k-r} + \cdots. \tag{6}$$

By the Staudt-Clausen theorem lb^{a} is an integer or a fraction whose denominator is prime to l. All the coefficients of lb^{a} in the right-hand member of (6), commencing with the third, are divisible by l^{a+1}. To prove this, we observe that

$$(b+l)^{l} \equiv b^{l} + l^{2}b^{l-1} \pmod{l^{3}}$$

and

$$(b+l)^{sl^{a}} \equiv b^{sl^{a}} + sl^{a+1}b^{sl^{a}-1} \pmod{l^{a+2}},$$

whence

$$(b+l)^{sl^{a}+2} \equiv b^{sl^{a}+2} + (sl^{a}+2)lb^{sl^{a}+1} + b^{sl^{a}}l^{2} \pmod{l^{a+2}},$$

and

$$\frac{(b+l)^{sl^{a}+2} - b^{sl^{a}+2}}{sl^{a}+2} \equiv lb^{sl^{a}+1} + \frac{b^{sl^{a}}l^{2}}{2} \pmod{l^{a+2}}. \tag{7}$$

The left-hand member of this relation is precisely the right-hand member of (6) if we expand and set $k = sl^{a}+1$. Setting b_a for b^{a} in (7), noting that $b_{sl^{a}} = 0$ for s odd, and comparing (6) with the new (7), the relation (5) is obtained. We then deduce from (3) and (4)

$$h \equiv \frac{\prod_{s} l(-1)^{(sl^{a}-1)/2} B_{(sl^{a}+1)/2}}{2^{\frac{1}{2}(l-3)}} \pmod{l^{a}}, \tag{8}$$

$s = 1, 3, \cdots, l-2.$

Kummer* has shown that

$$\frac{B_a}{a} \equiv (-1)^{k\mu} \frac{B_{a+k\mu}}{a+k\mu} \pmod{l},$$

where k is an integer and a is not a multiple of $\mu = (l-1)/2$. This gives

$$(-1)^{s\mu} \frac{B_{(sl+1)/2}}{s(l+1)/2} \equiv \frac{B_{(s+1)/2}}{(s+1)/2} \pmod{l},$$

and applying the latter relation to (8) for $a = 1$, we obtain Kummer's result to the effect that the necessary and sufficient condition that h be divisible by l is that one of the Bernoulli numbers B_s, $(s = 1, 2, \cdots, \frac{1}{2}(l-3))$ is divisible by l.

BALA, PA.,
November, 1918.

CORRECTIONS AND NOTE TO THE CAMBRIDGE COLLOQUIUM OF SEPTEMBER, 1916.

BY PROFESSOR G. C. EVANS.

1. *Corrections.* On page 35 in equation (9′) change γ to α. In all the formulas and equations following on page 35 change γ to β and β to α.

On page 37 in equation (12) change γ to β and β to α.

On page 39 in the second equation there should be an i as a factor of each of the last two terms of the integrand.

2. *Note to Art. 27, the Analogue of Green's Theorem.* The approach to the analogue of Green's theorem is clearer if made in the following way, and bears more relation to the development with which we are familiar in calculus. The meaning of equations (17) to (20) is perhaps not clear, as the equations stand. But the invariant $H_{\Phi_1\Phi_1'}$, defined by

$$(V_1 \times V_2) H_{\Phi_1\Phi_1'} = W_1 \times W_2', \tag{21}$$

which may be rewritten in the new forms

$$\begin{aligned} H_{\Phi_1\Phi_1'} &= (\beta \cdot W_1)(\beta \cdot W_1') + (\alpha \cdot W_1)(\alpha \cdot W_1') \\ &= -(\beta \cdot W_1)(\alpha \cdot W_2') + (\alpha \cdot W_1)(\beta \cdot W_2'), \end{aligned}$$

* L. c., vol. 41, p. 368.

as we see below, bears a noticeable resemblance to the scalar product of two gradients which is subjected to the familiar analysis of Green's theorem.

The forms just written, however, do not lend themselves to the customary integration by parts. Let us then in these formulas replace the part which refers to, say, W_1 and W_2 by functions φ_1 and φ_2 which depend on them (retaining meanwhile the quantities W_1' and W_2') and seek to write $H_{\Phi_1\Phi_1'}$ in the form

$$H_{\Phi_1\Phi_1'} = W_1' \cdot \nabla\varphi_1 + W_2' \cdot \nabla\varphi_2$$

or more generally, introducing an arbitrary point function $k(x, y, z)$, in the form

$$kH_{\Phi_1\Phi_1'} = W_1' \cdot \nabla\varphi_1 + W_2' \cdot \nabla\varphi_2. \tag{22}$$

Green's theorem is merely the result of integrating both sides of (22) over a three-dimensional region, and reducing the right-hand member by an integration by parts.

What conditions must φ_1, φ_2 satisfy to make possible the equation (22)? A short analysis shows that a necessary and sufficient condition is given by equation (17), which may be deduced directly, starting from this point of view.

The analysis follows. In the first place, by making use of the identities (see (8) page 34)

$$W_1 = V_1\alpha \cdot W_1 + V_2\beta \cdot W_1, \quad W_2' = V_1\alpha \cdot W_2' + V_2\beta \cdot W_2',$$

we have

$$W_1 \times W_2' = (V_1 \times V_2)\{(\alpha \cdot W_1)(\beta \cdot W_2') - (\beta \cdot W_1)(\alpha \cdot W_2')\},$$

whence, from the definition (21),

$$H_{\Phi_1\Phi_1'} = -(\beta \cdot W_1)(\alpha \cdot W_2') + (\alpha \cdot W_1)(\beta \cdot W_2').$$

By the conditions of isogeneity (10″), we get the other new form mentioned for $H_{\Phi_1\Phi_1'}$, as well as additional ones of the same character.

The equation just written, by comparison with (22), gives us the equation

$$k[(\alpha \cdot W_1)(\beta \cdot W_2') - (\beta \cdot W_1)(\alpha \cdot W_2')] = W_1' \cdot \nabla\varphi_1 + W_2' \cdot \nabla\varphi_2.$$

In the right-hand member the following substitutions may be

made, to make the two members similar in construction:

$$W_1' \cdot \nabla\varphi_1 = \nabla\varphi_1 \cdot (V_1\beta - V_2\alpha) \cdot W_2'$$
$$= (V_1 \cdot \nabla\varphi_1)(\beta \cdot W_2') - (V_2 \cdot \nabla\varphi_1)(\alpha \cdot W_2'),$$
$$W_2' \cdot \nabla\varphi_2 = \nabla\varphi_2 \cdot (V_1\alpha + V_2\beta) \cdot W_2'$$
$$= (V_1 \cdot \nabla\varphi_2)(\alpha \cdot W_2') + (V_2 \cdot \nabla\varphi_2)(\beta \cdot W_2').$$

Hence the equation becomes

$$(\nabla\varphi_1 \cdot V_1 + \nabla\varphi_2 \cdot V_2 - k\alpha \cdot W_1)(\beta \cdot W_2')$$
$$+ (\nabla\varphi_2 \cdot V_1 - \nabla\varphi_1 \cdot V_2 + k\beta \cdot W_1)(\alpha \cdot W_2') = 0.$$

By hypothesis, φ_1, φ_2, k are independent of W_1', W_2'; hence by putting $W_2' = V_2$ and V_1 successively the following equations are deduced

$$k\alpha \cdot W_1 = \nabla\varphi_1 \cdot V_1 + \nabla\varphi_2 \cdot V_2,$$
$$k\beta \cdot W_1 = \nabla\varphi_1 \cdot V_2 - \nabla\varphi_2 \cdot V_1,$$

which with the identity $W_1 = V_1\alpha \cdot W_1 + V_2\beta \cdot W_1$ yield the result

$$kW_1 = (V_1V_1 + V_2V_2) \cdot \nabla\varphi_1 + (V_1V_2 - V_2V_1) \cdot \nabla\varphi_2,$$

which is merely (17). Vice versa, from (17) may be deduced (22). Equation (18) is equivalent to (17) by the relation of isogeneity.

In a similar manner equation (19) or (20) is seen to be a necessary and sufficient condition for the functions φ_1' and φ_2' to satisfy in order that $H_{\varphi_1\varphi_1'}$ may be written in the form (22′).

Equation (17) is equivalent to two independent differential equations on the two functions φ_1, φ_2 to be determined, as is seen in the equivalence of that equation to the two preceding (i. e., the conditions in the plane V_1V_2). Hence the function k remains entirely arbitrary, and may if we choose be taken everywhere positive.

PARIS, FRANCE,
February, 1919.

CIRCLE AND SPHERE GEOMETRY.

A Treatise on the Circle and the Sphere. By JULIAN LOWELL COOLIDGE. Oxford, the Clarendon Press, 1916. 8vo. 603 pp.

AN era of systematizing and indexing produces naturally a desire for brevity. The excess of this desire is a disease, and its remedy must be treatises written, without diffuseness indeed, but with the generous temper which will find room and give proper exposition to true works of art. The circle and the sphere have furnished material for hundreds of students and writers, but their work, even the most valuable, has been hitherto too scattered for appraisement by the average student. Cyclopedias, while indispensable, are tantalizing. Circles and spheres have now been rescued from a state of dispersion by Dr. Coolidge's comprehensive lectures. Not a list of results, but a well digested account of theories and methods, sufficiently condensed by judicious choice of position and sequence, is what he has given us for leisurely study and enjoyment.

The reader whom the author has in mind might be, well enough, at the outset a college sophomore, or a teacher who has read somewhat beyond the geometry expected for entrance. Beyond the sixth chapter, however, or roughly the middle of the book, he must be ready for persevering study in specialized fields. The part most read is therefore likely to be the first four chapters, and a short survey of these is what we shall venture here.

After three pages of definitions and conventions the author calls "A truce to these preliminaries!" and plunges into inversion. Circles orthogonal to a fixed circle are self-inverse, and this property generalized defines anallagmatic curves. A family of such circles will have an anallagmatic envelope, for example. A novel term, deferrent, means the locus of centers of a family of orthogonal circles. In 35 closely linked theorems we advance to mutually tangent circles, and find at once a demonstration of Steiner's favorite theorem on poristic systems of circles, whose concise formulation warrants quotation (page 34): "Given two non-intersecting circles which possess the property that a ring of n circles may be

constructed all tangent to them and successively tangent to one another making m complete circuits, and if two circles of the ring touch the original ones at points on one circle orthogonal to these two, then the original circles are members of a ring of n_1 circles making m_1 complete circuits, all tangent to the two of the first ring, where

$$\frac{m}{n}+\frac{m_1}{n_1}=\frac{1}{2}.$$"

Proving soon after this a condition given by Casey for four mutually external circles tangent to a fifth, the author reduces the four to points upon the fifth circumference and so derives Ptolemy's theorem: "If a convex quadrilateral be inscribed in a circle, the sum of the products of the opposite sides is equal to the product of the diagonals," these lengths being the segments on common tangents employed in Casey's formula. The converses of both are established; then follows the nine-point circle.

A section on circles related to a triangle brings in the names of Euler, Simson, Barrow, Fontené, Nagel, Fuhrmann, Greiner, and Kantor. We cite one theorem whose lowest case is commonly given in textbooks (page 48). "If a polygon be inscribed in a circle and tangents be drawn at all of its vertices, the product of the distances of any point of the circle from these tangents is equal to the product of its distances from the side-lines."

Brocard points of a triangle, Lemoine's circles, Steiner's point, Tarry's point, and others, fill a twenty-page section on the geometry of the triangle (not so entitled, the author restricting his selection severely). Concurrent circles and concyclic points then claim attention,—Miquel's circle, and others due to Grace, Pesci, and our author himself (see *Annals of Mathematics*, series 2, volume 12). Coaxal systems of circles (18 pages) close the first and properly the longest chapter, on the circle in elementary plane geometry. We quote two of the shortest theorems. "The three circles on the diagonals of a complete quadrilateral as diameters are coaxal." This is ascribed to Gauss and Bodenmiller, 1830. "If a circle so moves that it cuts two others in diametrically opposite points, or cuts one in diametrically opposite points and the other orthogonally, it will generate a coaxal system" (page 108).

"Suggestions" are appended to this and to later chapters, such as this: "It seems likely that there are other simple criteria for various systems of tangent circles like Casey's condition for four circles tangent to a fifth, Vahlen's criterion for poristic systems, or the Euler conditions that there may be a triangle or a quadrilateral inscribed in one circle which is circumscribed to the other. . . . It seems likely that there are other chains of concurrent circles and concyclic points besides those noticed in theorems 162–6," etc. Comments of this kind, following protracted expositions reaching often a high degree of intrinsic beauty, are eminently in place.

Chapter 2 treats of the circle in cartesian plane geometry. A labored section on the treatment of circles by trilinear coordinates bears out the conclusion that those coordinates "are not adapted to dealing with the circle in any broad way," but "only in studying those properties of a circle which are related to a particular triangle." Tetracyclic coordinates are therefore introduced, so defined that if (xy) denotes the bilinear sum $\Sigma_1^4 x_i y_i$, points or zero circles have $(xx) = 0$. A neat form of identical relation among ten circles, ascribed either to Darboux or to Frobenius, is then much employed in specialized applications. It consists in the vanishing of a five-rowed determinant, obtained by separating the ten circles into two sets of five, denoted by x, y, z, s, t and x', y', z', s', t', each letter indicating of course four tetracyclic coordinates. The constituents of the determinant are then the 25 bilinear sums like

$$(xx'), (xy'), \ldots, (tx'), (ty'), \ldots, (tt').$$

The examples given, 21 in number, are all decidedly interesting.

Of greater interest is doubtless the fourth section of this chapter, on analytical systems. Some of Morley's memorable work figures here. A less known line of discussion is that on a general cubic series of circles (page 159 seq.). "The common orthogonal circles to corresponding triads in three projective pencils of circles whereof no two have a common member will generate a general cubic series, and every general cubic series may be so generated in ∞ ways." Also: "The locus of the centers of the circles of a general cubic series is a rational curve of the third order." Obviously the subject is the direct homotype of the twisted cubic curve in point space of three dimensions; but as incidence in space is orthogonality between

circles, this version is a possible source of fruitful suggestions. Congruences, or two-parameter families, receive due attention, and the field partly developed is recommended for further exploration.

Famous problems in construction, the third chapter, is unique in the full discussion it gives of each problem with respect to Lemoine's characters of simplicity and exactness. The chief problems are those of Apollonius (to construct a circle tangent to three given circles), of Steiner (to construct a circle meeting three given circles at given angles), Malfatti's (to construct three circles, each of which shall touch the other two, and two sides of a given triangle), and two of Fiedler's. The chapter closes with an account of Mascheroni's geometry of the compasses.

Of later chapters, on more special themes, we mention those on pentaspherical space, on circle transformations and sphere transformations, on the oriented circle (and Laguerre transformations); on algebraic systems of circles in space, and on oriented circles in space. In particular we commend to readers the "suggestions" (page 473) that close the chapter on circle crosses; and the paragraph (page 482) on the pentacycle of Stephanos, and that which follows it.

The book has excellent indices, and an added page of errata and addenda. As a whole, it is a welcome gift to American and English geometricians. It has in prospect a long career of usefulness as a compendium of accessible fact and as a source of stimulation and suggestion. Incidentally it affords the refreshing detached humor that marks it as authentic.

H. S. WHITE.

SHORTER NOTICES.

Unified Mathematics. By L. C. KARPINSKI, H. Y. BENEDICT and J. W. CALHOUN. New York, D. C. Heath and Company, 1918. viii + 522 pp.

THE increasing popularity of a unified course in mathematics for college freshmen is reflected in the large number of text books covering this field which have recently been published. Moreover, the movement towards the general adoption of

such a course will undoubtedly be much accelerated by the variety of books available. This movement is clearly a logical one and in harmony with the general tendency in educational practice to avoid useless repetition and to emphasize the coordination of related ideas. A properly unified course should make for a better understanding of the nature of mathematics and a more usable knowledge of the mathematical principles that have been studied.

The work under consideration here has many excellent features. One of its principal merits may be found in the interesting way in which the theories developed are illustrated by numerous applications which are of real importance in the world of to day. This particular characteristic of the book is all the more commendable in view of the fact that many mathematical texts are rather lacking in this respect. As a result, it is quite possible for students who have completed a first course in calculus to have a very inadequate idea of the large number of useful applications of the theories they have learned. This ought not to be the case; there is just as good reason for discussing the applications in a course on mathematical theory as in a course on chemical theory.

It is the firm conviction of the reviewer that teachers of mathematics can do much toward fortifying the position of their subject in general education by writing and using texts that lay stress on the modern applications of mathematical theory. In this way courses in mathematics will become richer in content and more interesting for the average student, and the critics of mathematics will be largely disarmed. "Unified Mathematics" furnishes an excellent illustration of what can be done in this direction by writers who are willing to make the effort.

We will now consider the book somewhat more in detail. It may be stated in the first place that the subject matter includes all the more important topics of algebra, trigonometry and analytical geometry that are ordinarily treated in freshman courses. Moreover, excellent use has been made of the relationship between these three fields and a real unification has been achieved. The arrangement of material is good both from a logical and a pedagogical standpoint. Throughout the book a proper emphasis is laid on computation and one of the earlier chapters is devoted to showing how the application of certain algebraic formulas to arithmetical com-

putations serves to diminish considerably the labor of the computer. The introduction of sets of "timing exercises" in cases where the problems to be solved are of a rather mechanical nature, is a novel feature and ought to help in developing greater facility in mathematical technique on the part of the student.

As stated above, the book is unusually rich in applications. A list of the more important and novel ones will readily illustrate this fact. A treatment of compound interest is given in connection with the chapter on exponents and logarithms. A discussion of annuities, bonds at premium and discount, and sinking funds is found in the chapter on geometrical progressions. Railroad curves are treated in the chapter on right triangles and the laws for reflection and refraction of light are stated and applied in connection with the chapter on oblique triangles. In the chapters on the ellipse and parabola the uses of these curves in architecture is explained and problems connected with elliptical and parabolic arches are introduced. In a special chapter on applications of conic sections we find discussions of Kepler's laws, projectiles, reflectors, and elliptical gears. In the chapter on wave motion there is a brief discussion of sound waves, light waves, and electrical waves; a treatment of piston rod motion is also included. In a chapter entitled "Laws of Growth" various applications of the exponential function in connection with physical and biological phenomena are treated. Among the topics discussed are the law of organic growth, Halley's law for the decrease in air pressure, damped vibrations, and the curve of healing of a wound.

A student that has been made acquainted with such a wide range of applications for the theories he has learned will certainly have a great deal more respect for the usefulness of mathematical knowledge. Moreover, his interest in the subject will undoubtedly be stimulated and he will be more apt to continue his work in mathematics after the freshman year. As such results are undoubtedly desired by all serious minded teachers of mathematics, "Unified Mathematics" ought to be widely used.

CHARLES N. MOORE.

Vorlesungen über Zahlen- und Funktionenlehre. Erster Band, erste Abteilung. By ALFRED PRINGSHEIM. Leipzig, B. G. Teubner, 1916.

THIS is the first part of the first volume of an introductory treatise on the theory of functions of real and complex variables. The author proposes to include in the first volume, in addition to a detailed logical treatment of the rational numbers, a complete exposition of the material treated by him in his two articles in the Encyklopädie: "Irrationalzahlen und Konvergenz unendlicher Prozesse" and "Unendliche Prozesse mit komplexen Termen," the only part omitted being the theory of infinite determinants. The part of the first volume under review concerns itself with the following principal topics: the system of real numbers, convergent sequences of such numbers, limits, orders of infinity and of infinitesimals, and double sequences of real numbers.

The book is intended primarily for students who are beginning the study of higher mathematics, and the theory is built up throughout from first principles. In fact the author declares in his preface that no previous mathematical knowledge is in general necessary for the comprehension of the first volume. In spite of the elementary character of the exposition, rigor of proof is nowhere sacrificed and the reader is made acquainted with the modern viewpoints in the domain of analysis that is treated.

CHARLES N. MOORE.

I Numeri reali e l'Equazione esponenziale $a^x = b$ per le Scuole Medie Superiori. By Dr. GAETANO FAZZARI. Palermo, Libreria Scientifica D. Capozzi, 1918. 75 pp. Lire 1.80.

THIS little book, as its title indicates, is devoted mainly to the theory of real numbers. It has no scientific pretense, as the author states in the short preface, but contains an exposé of the theory as Dr. Fazzari has presented it for several years to his students at the R. Liceo Umberto I in Palermo.

The book is divided into two parts. In part I is found the theory of real numbers, the development being essentially that of Dedekind. It commences with the definition of a class of rational numbers; i. e., a series of rational numbers satisfying a given condition. Of two classes A and A', A is said to be inferior to A', and A' superior to A, if each element of A is less than every element of A'.

There follows the notion of contiguity. Two classes are contiguous if one is inferior to the other and if it is possible to find an element of one and an element of the other such that their difference is less than any given rational number, not zero, arbitrarily small. This definition is indicated symbolically by writing (A, A') with the conditions $a' > a$ and $a_n' - a_m < \epsilon$.

A number α is the limit of two contiguous classes (A, A') if $a \leqq \alpha \leqq a'$, and this is denoted by writing $\alpha = (A, A')$.

There follows the existence theorem: There exist contiguous classes of rational numbers which do not have a rational limit. The proof is essentially that of Dedekind. The converse is a very simple statement of the Dedekind postulate: If two contiguous classes do not have a rational limit, we shall say that there is one number, and but one, called an irrational number, which is their limit.

The class of all rational numbers and all irrational numbers constitute the class of real numbers.

Sections 2 to 9 inclusive of part I deal with the conditions for the equivalence of two real numbers $\alpha = (A, A')$ and $\beta = (B, B')$, and with the arithmetic operations of addition, subtraction, multiplication, division, involution, and evolution as applied to real numbers defined by contiguous classes.

The first two sections of part II deal with the theory of indices, the exponent being any real number. The last section is devoted to the solution of the exponential equation $a^x = b$.

A series of exercises is inserted after each section, dealing with the subject matter of that section, and there are numerous examples throughout the book to illustrate definitions and theorems.

The treatment is remarkably clear, but one would expect, even in an elementary treatise, some reference to the sources from which the material is drawn. No mention is made, however, of Dedekind or of others who have worked in this field.

Students of mathematics in this country would find no especial difficulty in mastering the contents of Dr. Fazzari's book, but would be unappreciative of the necessity or the beauty of the theory, at least until they were quite beyond the elementary mathematics of our secondary schools.

L. WAYLAND DOWLING.

Infinitesimal Calculus. By F. S. CAREY. Longmans, Green and Company, 1917. Section 1, 144 pp.; Section 2, 207 pp.

THIS text is bound in two separate sections, the first containing sufficient material for a good elementary course, while the two sections together cover the topics usually presented in a longer elementary course. Chapter one reminds one of function theory, for the author treats of such topics as sequence of numbers, the arithmetical continuum, closed and open ranges, etc. The reviewer doubts if a beginner can grasp such concepts. Chapter two, on "Limits," contains many interesting examples to illustrate what a limit is, but nowhere is there to be found a concise definition of the word. The question of left hand and right hand limit, the question of the limit of a sum, product and quotient of two functions, is very thoroughly discussed. In the third chapter the rules for the derivative of a sum, a product and quotient of two functions are derived, but the last two derivations are very blind. The next chapter "The sign of the differential coefficient" treats of maxima and minima, and many fine examples are to be found among the exercises. This is followed by a chapter on algebraic functions. The remaining topics treated in the short elementary course are: "The inverse of a function," "Function of a function," "Tangent and normal," "Parametric equations," "Point of inflexion," "Circle of curvature," "Order of magnitude," "Inverse differentiation," "Logarithmic functions," "Areas," "Volume," "Parabolic approximation," "Simpson's rule," "Moments" and "Center of gravity." No definite integrals are used, in fact the symbol $\int$ is not introduced.

Section two starts with an excellent chapter on exponential and hyperbolic functions. The results of several integration formulas are expressed in terms of these functions. This is followed by a discussion of the motion of a particle along an axis. The definite integral is now introduced and many of the elementary properties which we usually assume in an elementary course are proved in full detail. This is followed by a chapter on polar coordinates in which pedal curves and intrinsic equations are discussed together with the usual material to be found in such a chapter. Work on partial differentiation, double integration, triple integration, expansion in power series, curve tracing, singular points, Newton's method of ascertaining the form of a curve at the origin and

at infinity, envelopes, involutes, roulettes and planimeters, finish the work in the calculus. This is followed by a short course in differential equations and a short but very interesting chapter on nomography.

The book is well written and the typographical errors are few, there being more, however, in the second section than in the first.

F. M. MORGAN.

Annuaire du Bureau des Longitudes pour l'An 1919. Paris, Gauthier-Villars.

THERE are few changes in the *Annuaire* for 1919 which call for special notice. Every year the editors have evidently to consider the pressure of matter for which there is a demand and the chief problem appears to be in finding out what can be omitted in order to make room for new data. The tendency seems to be in the direction of making the volume more useful to the person with scientific interests, rather than to the general reader, for we notice that articles on such subjects as coinage, legal measures, and geography are either suppressed or cut down.

An interesting summary of our present knowledge of the figures of the equilibrium of a liquid is furnished by M. Appell. The account is free from mathematical developments and it is written in a style which gives the main facts; there is an excellent list of the chief memoirs on the subject attached to the end of the article. This subject has been lately revived on account of the papers continuing Darwin's work in the *Transactions of the Royal Society* by Jeans. M. Hamy, in a second appendix, discusses the possibilities of the determination of the actual dimensions of stars by interference methods. He believes that some indications of value can be obtained in this way in spite of the extremely minute angles that the diameters of even the nearest stars subtend at the earth.

E. W. BROWN.

NOTES.

THE twenty-sixth summer meeting of the American Mathematical Society will be held at the University of Michigan, Ann Arbor, Mich., on Tuesday, Wednesday, and Thursday, September 2–4, 1919, beginning on Tuesday afternoon at two o'clock. The meeting will be followed by that of the Mathematical Association of America on September 4–5. A joint session of the two organizations is planned for Thursday afternoon, and a joint dinner for Thursday evening. Abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary by August 12 if it is desired that they should be printed in advance of the meeting; otherwise they may be sent in by August 18.

THE committee on arrangements for the summer meeting and colloquium of the Society to be held at the University of Chicago in 1920 announce the following lecturers and provisional subjects: Professor G. D. BIRKHOFF: "Dynamical systems;" Professor F. R. MOULTON: "Certain topics in functions of infinitely many variables."

AMONG the subjects to be considered at the series of scientific conferences to be held at Brussels, beginning July 18, is the formation of an international mathematical union. Such unions have already been organized in astronomy, chemistry, and geophysics. Whatever tentative plans may evolve for a mathematical union will be submitted to the various mathematical organizations of the world for discussion.

THE General education board has appropriated $16,000 for salaries and other expenses of the National committee on mathematical requirements. Chairman J. W. YOUNG, as representative of the colleges, and vice-chairman J. A. FOBERG, as representative of the secondary schools, will devote their whole time during the coming year to the work of the committee, which will cover the revision of secondary school and college courses and of college entrance requirements in mathematics.

THE technical staff of the U. S. ordnance department has been authorized to secure the services of five experts in mathematics and dynamics, at salaries ranging from $2,500 to $5,000, to conduct scientific research on ordnance problems, act as advisers on all mathematical and scientific problems for the ordnance deparment, and keep up connections between the department and the scientific world.

AT the meeting of the London mathematical society held April 24, the following papers were read: By K. A. RAU, "Lambert's series" and "The relations between the convergence of a series and its summability by Cesàro means"; by G. H. HARDY and J. E. LITTLEWOOD, "Tauberian theorem for Lambert's series"; by W. H. YOUNG, "A formula for an area."

At the meeting of May 15 the following papers were read: By G. N. WATSON, "Zeros of Lommel's polynomials"; by W. H. YOUNG, "The triangulation method of defining the area of a surface."

AT the meeting of the Edinburgh mathematical society on June 13 the following papers were read: By W. L. MARR, "Systems of apolar triads in the twisted cubic" and "The co-symmedian system of tetrahedra in a sphere"; by D. M. Y. SOMMERVILLE, "Approximations to the length of an arc"; by D. G. TAYLOR, "A new approximate formula for the complete elliptic integrals $K(k)$, $E(k)$."

THE Association of teachers of mathematics in the middle states and Maryland held a joint meeting with the Association of mathematics teachers of New Jersey at the Central high school, Newark, on May 3. Papers were read by Professor A. L. JONES, Mr. H. E. WEBB, Professor H. B. FINE, and Mr. P. C. H. PAPPS.

THE following university courses in mathematics are announced:

JOHNS HOPKINS UNIVERSITY (academic year 1919–1920).—By Professor FRANK MORLEY: Higher geometry, three hours (first term); Theory of functions, three hours (second term); Seminar, two hours.—By Professor L. S. HULBURT: Projective geometry and higher plane curves, three hours.—By

Professor A. COHEN: Elementary theory of functions, two hours; Differential equations and mechanics, three hours.—By Dr. F. D. MURNAGHAN: Electricity and magnetism, three hours (first term); Dynamics of a rigid body, three hours (second term).

UNIVERSITY OF CALIFORNIA (academic year 1919–1920).—By Professor M. W. HASKELL: Higher plane curves, three hours (first term); Advanced analytic geometry, three hours (second term).—Professor C. A. NOBLE: Functions of a complex variable, three hours; Elementary mathematics for advanced students, three hours (first term).—By Professor D. N. LEHMER: Theory of numbers, three hours (first term); Algebraic surfaces, three hours (second term).—By Professor FLORIAN CAJORI: History of mathematics, two hours; History of physics, two hours (second term); Seminar, two hours; Teaching of mathematics, three hours (first term).—By Professor T. M. PUTNAM: Solid analytic geometry, three hours (first term); Theory of algebraic equations, three hours (second term).—By Professor FRANK IRWIN: Advanced calculus, three hours.—By Professor B. A. BERNSTEIN: Algebra of logic, three hours (first term); Theory of probability, three hours (second term); Logic of mathematics, two hours.—By Professor THOMAS BUCK: Advanced calculus, two hours.—By Professor J. H. MCDONALD: Analytic mechanics, three hours; Partial differential equations, three hours (first term).

UNIVERSITY OF STRASBOURG (second semester, 1919).—By Professor ESCLANGON: Astronomy.—By Professor M. FRÉCHET: General analysis; Advanced calculus.—By Professor E. GAU: Partial differential equations of the second order; Elementary calculus.—By Professor VERONNET: Astronomy.—By Professor VILLAT: Aerodynamics; Mechanics.

AT the celebration, on September 21, 1918, of the 250th anniversary of the founding of the University of Lund, honorary doctorates were conferred on GUSTAF ENESTRÖM, editor of *Bibliotheca Mathematica*, and on J. L. W. V. JENSEN, engineer and member of the editorial board of *Acta Mathematica*.

THE Royal academy of sciences of Denmark offers its gold medal for a memoir on the following subject: "To lay the

foundations of a general theory of surfaces not everywhere convex, projectively closed, and without double lines. It is desirable to reduce as much as possible the hypotheses relative to the infinitesimal properties of the surfaces." Competing memoirs may be written in Danish, Norwegian, Swedish, English, German, French or Latin, and should be sent to Professor KNUDSEN, of the Polytechnic school of Copenhagen, before October 31, 1919.

THE Ackermann-Teubner prize for 1918 was awarded to Professor L. PRANDTL, of Göttingen, for his memoir "Ueber den Luftwiderstand von Kugeln" in the *Göttinger Nachrichten* for 1914.

DR. W. GROSS, of the University of Vienna, has been promoted to a professorship of mathematics.

PROFESSOR E. COSSERAT, of the University of Toulouse, has been elected non-resident member of the Paris Academy of sciences.

PROFESSOR E. GOURSAT, of the University of Paris, has been elected member of the Paris academy of sciences, to succeed Professor E. PICARD who has been chosen perpetual secretary of the academy.

PROFESSOR E. PASCAL, of the University of Naples, and Professor E. ALMANSI, of the University of Rome, have been elected national members of the Reale Accademia dei Lincei, of Rome. Professor G. FANO, of the University of Turin, has been elected a corresponding member.

PROFESSOR G. N. WATSON, of the University of Birmingham, has been elected fellow of the Royal society of London.

PROFESSOR WILLIAM BURNSIDE, of the Royal naval college, Greenwich, has retired from teaching.

PROFESSOR JACQUES HADAMARD, of the Collège de France, has accepted an invitation from Yale University to be a Silliman lecturer in the spring of 1920.

MAJOR J. L. COOLIDGE, of Harvard University, gave a course of two lectures a week at the University of Paris on "Geometry in the complex domain" during the spring term.

AT the University of California, Professor VITO VOLTERRA, of the University of Rome, will deliver a series of six lectures in August or September. Professor FRANK MORLEY, of Johns Hopkins University, will be head of the department of mathematics for the summer session, and will conduct the mathematical seminar.

PROFESSOR O. D. KELLOGG, of the University of Missouri, has been made a fellow of the American Association for the Advancement of Science, and elected vice-president and chairman of section A.

PROFESSOR E. B. VAN VLECK, of the University of Wisconsin, has accepted an invitation as lecturer on mathematics at Harvard University for the second half of the ensuing academic year. Professor Van Vleck will give, besides a course in the calculus, one on the theory of functions of a complex variable and one on the partial differential equations of mathematical physics.

PROFESSOR M. I. PUPIN, of Columbia University, until recently consul general for Serbia to the United States, has been attending the peace conference at Paris.

PROFESSOR G. C. EVANS, who has been in service as scientific attaché to the American Embassy at Rome, has returned to Rice Institute.

PROFESSOR F. R. SHARPE, of Cornell University, has been promoted to a full professorship of mathematics. Mr. V. C. GROVE has been appointed instructor in mathematics.

AT the Sheffield scientific school, Yale University, the following instructors in mathematics have been appointed: Mr. HERMAN BETZ, Dr. E. A. T. KIRCHER, Dr. J. M. STETSON, and Mr. EUGENE TAYLOR.

IN the mathematics department of the University of Illinois, Dr. J. R. KLINE, of the Sheffield scientific school, has been appointed associate and Dr. C. C. CAMP has been appointed instructor.

DR. C. A. FISCHER, of Columbia University, has been appointed Seabury professor of mathematics and astronomy at Trinity College, Hartford.

AT the University of Colorado assistant professor G. H. LIGHT has been promoted to an associate professorship of mathematics.

MR. C. A. NELSON has been appointed instructor in mathematics at the University of Kansas.

MR. A. D. CAMPBELL, of Cornell University, has been appointed instructor in mathematics in Yale University.

AT Brown University, Captain R. E. GILMAN, of Cornell University, and Captain R. W. BURGESS have been appointed assistant professors of mathematics, and Mr. C. R. ADAMS instructor in mathematics.

PROFESSOR E. G. BILL, who has been serving as director of the military service branch at Ottawa, Canada, will return to Dartmouth College in the fall.

PROFESSOR P. P. BOYD, of the University of Kentucky, has been elected president of the Kentucky academy of science.

PROFESSOR E. D. ROE, Jr., of Syracuse University, has been elected director of the observatory. His position in the department of mathematics remains unchanged.

CAPTAIN A. L. UNDERHILL, of the University of Minnesota, has been appointed commandant at the University of Grenoble, where several hundred American soldiers are studying.

DR. H. C. M. MORSE and Dr. I. A. BARNETT have been appointed Benjamin Peirce instructors in Harvard University.

MR. RAYMOND DOUGLAS of the University of Maine, Dr. J. S. TAYLOR, of the University of California, and Dr. NORBERT WIENER have been appointed instructors in mathematics in the Massachussetts Institute of Technology.

DR. H. L. SLOBIN, of the University of Minnesota, has been appointed head of the department of mathematics in New Hampshire College.

MR. H. M. SHOWMAN, of the Colorado School of Mines, has been appointed assistant professor of mathematics at the Case school of applied science.

PROFESSOR E. E. MOOTS, of Cornell College, Iowa, has been elected head of the department of mathematics and engineering.

MR. P. W. HILL, who has been in service over seas, will resume his instructorship in mathematics at Wabash College.

MR. H. L. SMITH and Mr. WARREN WEAVER have been appointed instructors in mathematics at the University of Wisconsin.

MISS HELEN BARTON, of Salem College, and Miss MARION E. STARK, of Meredith College, have been appointed instructors in mathematics at Wellesley College.

DR. MARION B. WHITE, of the Michigan State normal college, Ypsilanti, has been appointed professor of mathematics at Carleton College.

The death is announced, on February 27, 1919, of Professor A. M. LIAPOUNOFF, of the University of Petrograd.

PROFESSOR M. B. WEINSTEIN died recently at Berlin in his sixty-fifth year.

MR. ROBERT BOWES, of the firm of Bowes and Bowes, Cambridge, publishers of the *Messenger of Mathematics*, died February 9, 1919, in his eighty-fourth year.

PROFESSOR A. B. NELSON, of Central University, Danville, Ky., died in November, 1918.

BOOK CATALOGUES: Cambridge University Press, list of publications, including over 200 titles in mathematics, physics and chemistry; Librairie Paul Ritti, Paris, list of 45 titles in mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BIGOURDAN (G.). Les premières sociétés savantes de Paris au XVIIe siècle et les origines de l'Académie des Sciences. Paris, Gauthier-Villars, 1918. 4to. 24 pp. Fr. 1.50

CASHMORE (M.). Fermat's last theorem: three proofs by elementary algebra. Revised edition. London, Bell, 1919. 55 pp. 2s. 6d.

DICKSON (L. E.). History of the theory of numbers. Volume 1: Divisibility and primality. Washington, Carnegie Institution, 1919. Sm. 4to. 12 + 486 pp. Paper $7.50; cloth, $8.00

MERCOGLIANO (D.). See YOUNG (J. W.).

MORÁVEK (G.). Allgemeine Beweise der Gültigkeit des letzten Fermatschen Satzes über die unbestimmte Gleichung $z^n = x^n + y^n$. Mit Ergänzungen. Prag, G. Morávek, 1917. 8vo. 2 pp.

—— Allgemeine Beweise der Gültigkeit des letzten Fermatschen Satzes. Prag, G. Morávek, 1918. 1 p.

RUSSELL (B.). Introduction to mathematical philosophy. London, Allen and Unwin, and New York, Macmillan, 1919. 8vo. $3.00

SCHLÜTER (H.). Die höhere Mathematik als allgemeinverständliches Rechnungsmittel. Berlin, Meusser, 1917. 8vo. 50 pp.

VILLANI (N.). L'equazione di Fermat $x^n + y^n = z^n$, con dimostrazione generale. Lanciano, G. Carabba, 1918. 8vo. 17 pp. L. 2.00

YOUNG (J. W.). I concetti fondamentali dell'algebra e della geometria. Versione e note di Domenico Mercogliano. (Manuali Pierro.) Napoli, 1919. L. 8.00

II. ELEMENTARY MATHEMATICS.

BOURDON (P. L. M.). Trigonométrie rectiligne et sphérique. Edition revue et annotée par C. Brisse. Nouveau tirage. Paris, Gauthier-Villars, 1918. 8vo. Fr. 5.00

BRISSE (C.). See BOURDON (P. L. M.).

CAMINATI (C.) e CAMINATI (P.). Nuovo manuale italiano logaritmico-trigonometrico con sette o con dieci decimali. Piacenza, V. Porta, 1918. 8vo. 63 pp. L. 3.50

CAMINATI (P.). See CAMINATI (C.).

GÉRARD (L.). See NIEWENGLOWSKI (B.)

GOYEN (P.). Elementary mensuration. Constructive plane geometry and numerical trigonometry. London, Macmillan, 1919. 8 + 169 pp. 3s. 6d.

HARANG (F.). Eléments de trigonométrie suivis d'une instruction sur la règle à calcul. Paris, Dunod et Pinat, 1919. 8vo. 6 + 138 pp. Fr. 7.50

MACCAFERRI (E.). Calcolo numerico approssimato. Milano, Hoepli, 1919. 18mo. 16 + 200 pp. L. 5.50

MILNE (R. M.). Mathematical papers for admission into the Royal military college, and papers in elementary engineering for naval cadetships, July and November, 1918. Edited by R. M. Milne. London, Macmillan, 1919. 36 pp. 1s. 3d.

NIEWENGLOWSKI (B.) et GÉRARD (L.). Leçons de géométrie élémentaire à l'usage des classes de première C et D et de mathématiques A et B (programmes de 1905). I: Géométrie plane. 2e édition. Paris, Gauthier-Villars, 1918. 8vo. 4 + 354 pp. Fr. 7.00

PINCHERLE (S.). Geometria metrica e trigonometria. 8a edizione. Milano, Hoepli, 1918. 24mo. 8 + 160 pp. L. 3.00

SMITH (D. E.). Number stories of long ago. Boston, Ginn, 1919. 8 + 136 pp. $0.48

STONE (J. C.). Junior high school mathematics. Chicago, Sanborn, 1919. 12mo. 214 pp. $0.72

III. APPLIED MATHEMATICS.

APPUNTI di statica grafica. (R. Scuola di Applicazione per gli Ingegneri, Padova.) Padova, la Litotipo, 1918. 8vo. 252 pp. + 3 tavole. L. 13.00

ARIÈS (E.). Sur les tensions de vaporisation des liquides purs. (Thermodynamique.) Paris, Gauthier-Villars, 1918. 4to. 44 pp. Fr. 3.00

BARCELLONI (F.). See LORI (F.).

BAZARD (A.). Cours de mécanique. Tome 3: Hydraulique. Nouvelle édition revue, corrigée et augmentée. Paris, A. Michel, 1918. Royal 8vo. 600 pp. Fr. 18.00

BEAUFILS (H.). See HARANG (F.).

BÉRÉHARE (E.). See MARCOTTE (E.).

BIGOURDAN (G.). Histoire de l'astronomie d'observation et des observatoires en France. 1re partie: De l'origine à la fondation de l'observatoire de Paris. Paris, Gauthier-Villars, 1918. 4to. 184 pp. Fr. 12.00

BOUSSINESQ (J.). Complément à un récent mémoire des *Annales scientifiques de l'Ecole Normale Supérieure*, Sur la poussée des terres et l'état ébouleux, avec quelques idées générales sur la mécanique des semi-fluides et application de ces idées aux corps plastiques. Paris, Gauthier-Villars, 1918. 4to. 120 pp. Fr. 7.00

BRUZZÈSE (C.). Néo-mécanique, ou une révolution des bases de la mécanique. Torino, G. Fedetto, 1918. 8vo. 21 pp. L. 5.00

CAMERANO (G.). Manuale e prontuario del fresatore meccanico: raccolta di formule facili e relative tabelle indispensabili per la costruzione di viti e ruote dentate sulle fresatrici semplici e universali. Rocca S. Casciano, L. Cappelli, 1918. 8vo. 88 pp. L. 3.00

DESGARDES (E.). Calcul des ressorts. Formules pratiques et barèmes. Paris, Dunod et Pinat, 1919. 8vo. 75 pp. Fr. 6.00

EUCLIDIS opera omnia. Edit. J. L. Heiberg et H. Menge. Volume 8: Euclidis phaenomena et scripta musica, edit. H. Menge. Fragmenta,

collegit et disposuit J. L. Heiberg. (Bibliotheca scriptorum graecorum et romanorum Teubneriana.) Leipzig, Teubner, 1916. 12mo. 54 + 292 pp. Geb. M. 6.60

FORD (L. R.). Elementary mathematics for field artillery. Louisville, Ky., 1919. 8vo. 72 pp.

FRANCESCA (A. DELLA). See LORI (F.).

GIORLI (E.). Il meccanico, per uso dei capitecnici, capimeccanici, disegnatori, chauffeurs, aviatori, elettricisti, conduttori di caldaie a vapore, delle scuole per macchinisti ferrovie, scuole professionali, industriali e d'arti e mestieri. 8a edizione ampliata. (Manuali Hoepli.) Milano, Hoepli, 1919. 24mo. 15 + 555 pp. L. 9.50

GOODENOUGH (G. A.). See WATERBURY (L. A.).

HARANG (F.) et BEAUFILS (H.). Notions élémentaires de géométrie descriptive appliquée au dessin. Paris, Dunod et Pinat, 1919. 12mo. 188 pp. Fr. 5.40

HEIBERG (J. L.). See EUCLIDIS.

HIGBIE (H. H.). See WATERBURY (L. A.).

HJELMSLEV (J.). Laerebog i Geometri til Brug ved den polytekniske Laereanstalt. København, Gjellerup, 1918. 303 pp.

LEWIS (W. C. McC.). A system of physical chemistry. Volume 3: Quantum theory. New York, Longmans, 1919. 8vo. 8 + 209 pp. $2.50

LORI (F.). Elettrologia ed elettrotecnica: lezioni raccolte e redatte per cura degli studenti F. Barcelloni e A. Della Francesca. (R. Scuola di Applicazione per gli Ingegneri, Padova.) Padova, litotip. Universitaria, 1918. 8vo. 599 pp. L. 23.00

MARCOLONGO (R.). Meccanica razionale. Volume 2: Dinamica-meccanica dei sistemi deformabili. 2a edizione riveduta ed ampliata. (Manuali Hoepli.) Milano, Hoepli, 1918. 24mo. 13 + 413 pp. L. 6.00

MARCOTTE (E.) et BÉRÉHARE (E.). Résumé des connaissances scientifiques utiles aux aviateurs et mécaniciens de l'aéronautique. Paris, Dunod et Pinat, 1918. 8vo. 588 pp. Fr. 21.60

MENGE (H.). See EUCLIDIS.

MORELLI (E.). Costruzioni elettromeccaniche: calcolo, disegno e fabbricazione delle macchine elettriche, accessori ed applicazioni. Volume 1: Generatrici e corrente continue. 2a edizione, interamente riveduta. Torino, Unione tipografico-editrice, 1919. 8vo. 831 pp. + 31 tavole. L. 40.00

PORSTMANN (W.). Normenlehre, Grundlagen, Reform und Organisation der Mass- und Normensysteme. Leipzig, Schulwissenschaftlicher Verlag, 1917. 256 pp. M. 6.00

WATERBURY (L. A.), GOODENOUGH (G. A.) and HIGBIE (H. H.). A vest-pocket handbook for engineers. 3d edition. New York, Wiley, 1918. 16mo. 13 + 278 pp. $1.50

WHIPPLE (G. C.). Vital statistics. New York, Wiley, 1919. 16mo. 12 + 517 pp. $4.00

TWENTY-EIGHTH ANNUAL LIST OF PAPERS.

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

ALGER, P. L. Practical exterior ballistics. Read Dec. 27, 1918. *American Mathematical Monthly*, vol. 26, No. 3, pp. 101–102; March, 1919.

ALTSHILLER, N. On the *I*-centers of a triangle. Read (Southwestern Section) Dec. 1, 1917. *American Mathematical Monthly*, vol. 25, No. 6, pp. 241–246; June, 1918.

—— On the Teixeira construction of the unicursal cubic. Read (Southwestern Section) Dec. 1, 1917. *Annals of Mathematics*, ser. 2, vol. 20, No. 1, pp. 9–12; Sept., 1918.

BAUER, G. N., and SLOBIN, H. L. A system of algebraic and transcendental equations. Read Dec. 22, 1916. *American Mathematical Monthly*, vol. 25, No. 10, pp. 435–440; Dec., 1918.

BELL, E. T. A partial isomorph of trigonometry. Read (San Francisco) Dec. 14, 1918. *Bulletin of the American Mathematical Society*, vol. 25, No. 7, pp. 311–321; April, 1919.

BENNETT, A. A. Some mathematical features of ballistics. Read Dec. 27, 1918. *American Mathematical Monthly*, vol. 26, No. 3, p. 98; March, 1919.

BLICHFELDT, H. F. On low velocity high angle fire. Read Dec. 27, 1918. *American Mathematical Monthly*, vol. 26, No. 3, p. 103; March, 1919.

—— Report on the theory of the geometry of numbers. Read March 28, 1919. *Bulletin of the American Mathematical Society*, vol. 25, No. 10, pp. 449–453; July, 1919.

BLISS, G. A. The problem of Mayer with variable end points. Read Dec. 22, 1916. *Transactions of the American Mathematical Society*, vol. 19, No. 3, pp. 305–314; July, 1918.

—— Solutions of differential equations as functions of the constants of integration. Read Dec. 29, 1917. *Bulletin of the American Mathematical Society*, vol. 25, No. 1, pp. 15–26; Oct., 1918.

—— A method of computing differential corrections for a trajectory. Read March 29, 1919. *Journal of the United States Artillery*, vol. 50, No. 4, pp. 455–460; June, 1919.

BLUMBERG, H. A theorem on exhaustible sets connected with developments of positive real numbers. Read Dec. 27, 1913. *Annals of Mathematics*, ser. 2, vol. 20, No. 2, pp. 136–141; Dec., 1918.

—— On convex functions. Read (Southwestern Section) Dec. 2, 1916. *Transactions of the American Mathematical Society*, vol. 20, No. 1, pp. 40–44; Jan., 1919.

—— A theorem on linear point sets. Read Dec. 28, 1918. *Bulletin of the American Mathematical Society*, vol. 25, No. 8, pp. 350–353; May, 1919.

BUCHANAN, D. Orbits asymptotic to an isosceles-triangle solution of the problem of three bodies. Read Dec. 27, 1916. *Proceedings of the London Mathematical Society*, ser. 2, vol. 17, No. 1, pp. 54–74; June, 1918.

—— Asymptotic satellites near the straight-line equilibrium points in the problem of three bodies. Read Dec. 27, 1916. *American Journal of Mathematics*, vol. 41, No. 2, pp. 79–110; April, 1919.

CAJORI, F. On the history of "Playfair's parallel-postulate." Read (San Francisco) Dec. 14, 1918. *School Science and Mathematics*, vol. 18, No. 9, pp. 778–779; Dec., 1918.

—— A Greek tract on indivisible lines. Read (San Francisco) Dec. 14, 1918. *Science*, new ser., vol. 48, No. 1249, pp. 577–578; Dec. 6, 1918.

CARMICHAEL, R. D. Repeated solutions of a certain class of linear functional equations. Read Dec. 29, 1917. *Tôhoku Mathematical Journal*, vol. 13, No. 4, pp. 304–313; June, 1918.

—— General aspects of the theory of summable series. Read April 12, 1918. *Bulletin of the American Mathematical Society*, vol. 25, No. 3, pp. 97–131; Dec., 1918.

—— Fermat numbers $F_n = 2^{2^n} + 1$. Read (Southwestern Section) Dec. 1, 1917. *American Mathematical Monthly*, vol. 26, No. 4, pp. 137–146; April, 1919.

CHITTENDEN, E. W. On the Heine-Borel property in the theory of abstract sets. Read Oct. 26, 1918. *Bulletin of the American Mathematical Society*, vol. 25, No. 2, pp. 60–65; Nov., 1918.

—— On the limit functions of sequences of continuous functions converging relatively uniformly. Read Dec. 28, 1917. *Transactions of the American Mathematical Society*, vol. 20, No. 2, pp. 179–184; April, 1919.

COBLE, A. B. Theta modular groups determined by point sets. Read Sept. 4, 1917. *American Journal of Mathematics*, vol. 40, No. 4, pp. 317–340; Oct., 1918.

COLE, F. N. See WHITE, H. S.

COOLIDGE, J. L. The characteristic numbers of a real algebraic plane curve. Read Sept. 4, 1916. *Rendiconti del Circolo Matematico di Palermo*, vol. 42, Nos. 2–3, pp. 260–266; May–Dec., 1917.

CUMMINGS, L. D. The trains for the 36 groupless triad systems on 15 elements. Read Sept. 5, 1918. *Bulletin of the American Mathematical Society*, vol. 25, No. 7, pp. 321–324; April, 1919.

—— See WHITE, H. S.

CURTIS, M. F. The existence of the functions of the elliptic cylinder. Read Sept. 5, 1917. *Annals of Mathematics*, ser. 2, vol. 20, No. 1, pp. 23–34; Sept., 1918.

—— On the rectifiability of a twisted cubic. Read April 27, 1918. *Bulletin of the American Mathematical Society*, vol. 25, No. 2, pp. 87–88; Nov., 1918.

DANIELL, P. J. A general form of integral. Read (Southwestern Section) Dec. 2, 1916. *Annals of Mathematics*, ser. 2, vol. 19, No. 4, pp. 279–294; June, 1918.

—— Differentiation with respect to a function of limited variation. Read (Southwestern Section) Dec. 2, 1916 and Sept. 4, 1918. *Transactions of the American Mathematical Society*, vol. 19, No. 4, pp. 353–362; Oct., 1918.

—— A general form of Green's theorem. Read (Southwestern Section) Dec. 2, 1916. *Bulletin of the American Mathematical Society*, vol. 25, No. 8, pp. 353–357; May, 1919.

DICKSON, L. E. Mathematics in war perspective. Read Dec. 27, 1918. *Bulletin of the American Mathematical Society*, vol. 25, No. 7, pp. 289–311; April, 1919.

—— History of the theory of numbers. Volume 1: Divisibility and primality. Read April 3, 1915, Dec. 22, 1916 and (San Francisco) Oct. 27, 1917. Washington, Carnegie Institution, 1919. 12 + 486 pp.

—— Applications of the geometry of numbers to algebraic numbers. Read March 28, 1919. *Bulletin of the American Mathematical Society*, vol. 25, No. 10, pp. 453–455; July, 1919.

DINES, L. L. Projective transformations in function space. Read Dec. 27, 1916 and Sept. 4, 1917. *Transactions of the American Mathematical Society*, vol. 20, No. 1, pp. 45–65; Jan., 1919.

—— Systems of linear inequalities. Read March 29, 1919. *Annals of Mathematics*, ser. 2, vol. 20, No. 3, pp. 191–199; March, 1919.

EMCH, A. Proof of Pohlke's theorem and its generalizations by affinity. Read April 13, 1918. *American Journal of Mathematics*, vol. 40, No. 4, pp. 366–374; Oct., 1918.

—— On plane algebraic curves with a given system of foci. Read April 27, 1918. *Bulletin of the American Mathematical Society*, vol. 25, No. 4, pp. 157–161; Jan., 1919.

—— On closed curves described by a spherical pendulum. Read April 26, 1919. *Proceedings of the National Academy of Sciences*, vol. 4, No. 8, pp. 218–221; Aug., 1918. *Tôhoku Mathematical Journal*, vol. 15, Nos. 1–2, pp. 146–165; March, 1919.

—— On a certain generation of rational circular and isotropic curves. Read Dec. 29, 1917. *Bulletin of the American Mathematical Society*, vol. 25, No. 9, pp. 397–404; June, 1919.

EVANS, G. C. Functionals and their applications. Selected topics, including integral equations. Read Dec. 28, 1915, April 29, 1916, Sept. 6–8, 1916 and (Southwestern Section) Dec. 2, 1916. The Cambridge Colloquium, 1916, Part I, 12 + 136 pp.; New York, 1918.

FITE, W. B. Concerning the zeros of the solutions of certain differential equations. Read April 28, 1917, Oct. 27, 1917 and Dec. 27, 1917. *Transactions of the American Mathematical Society*, vol. 19, No. 4, pp. 341–352; Oct., 1918.

FORD, W. B. A conspectus of the modern theory of divergent series. Read Dec. 28, 1917. *Bulletin of the American Mathematical Society*, vol. 25, No. 1, pp. 1–15; Oct., 1918.

GARRETSON, W. V. N. On the asymptotic solution of the non-homogeneous linear differential equation of the nth order. A particular solution. Read Sept. 5, 1916. *American Journal of Mathematics*, vol. 40, No. 4, pp. 341–350; Oct., 1918.

GILLESPIE, D. C. Repeated integrals. Read Sept. 5, 1917. *Annals of Mathematics*, ser. 2, vol. 20, No. 3, pp. 224–228; March, 1919.

GLENN, O. E. Invariants which are functions of parameters of the transformation. Read Oct. 27, 1917. *Annals of Mathematics*, ser. 2, vol. 20, No. 2, pp. 125–135; Dec., 1918.

—— Modular concomitant scales, with a fundamental system of formal covariants, modulo 3, of the binary quadratic. Read April 27, 1918. *Transactions of the American Mathematical Society*, vol. 20, No. 2, pp. 154–168; April, 1919.

—— Covariants of binary modular groups. Read April 26, 1919. *Proceedings of the National Academy of Sciences*, vol. 5, No. 4, pp. 107–110; April, 1919.

GOULD, A. B. Notes concerning recent books on navigation. Read Dec. 27, 1918. *American Mathematical Monthly*, vol. 26, No. 3, p. 99; March, 1919.

GREEN, G. M. Memoir on the general theory of surfaces and rectilinear congruences. Read Sept. 5, 1816 and Dec. 27, 1917. *Transactions of the American Mathematical Society*, vol. 20, No. 2, pp. 79–153; April, 1919.

GROAT, B. F. Ice diversion, hydraulic models, and hydraulic similarity. Read Sept. 4, 1918. *Transactions of the American Society of Civil Engineers*, vol. 82, pp. 1138–1190; Dec., 1918.

GRONWALL, T. H. A theorem on power series, with an application to conformal mapping. Read Dec. 28, 1918. *Proceedings of the National Academy of Sciences*, vol. 5, No. 1, pp. 22–24; Jan., 1919.

HALLETT, G. H., JR. Concerning the definition of a simple continuous arc. Read Oct. 26, 1918. *Bulletin of the American Mathematical Society*, vol. 25, No. 7, pp. 325–326; April, 1919.

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